№ 13 (68) 2009
Выпуск 17/1
НАУЧНЫЙ РЕЦЕНЗИРУЕМЫЙ ЖУРНАЛ

Основан в 1995 г.

НАУЧНЫЕ ВЕДОМОСТИ Белгородского государственного университета
Математика Физика

Belgorod State University Scientific bulletin
Mathematics Physics

Учредитель:
Государственное образовательное учреждение высшего профессионального образования «Белгородский государственный университет»

Издатель:
Белгородский государственный университет.
Издательство БелГУ

Журнал зарегистрирован
в Федеральной службе по надзору за соблюдением законодательства
в сфере массовых коммуникаций
и охраны культурного наследия

Свидетельство о регистрации средства массовой информации ПИ № ФС77-21121 от 19 мая 2005 г.

СОДЕРЖАНИЕ

Введение. **5**

Вычеты логарифмических дифференциальных форм.
А.Г. Александров **6**

Логарифмические дифференциальные формы и комплексные динамические системы.
А.Г. Александров, А.А. Кастро, мл., В.А. Грузман **20**

О некторых разложениях $\zeta(3)$ в непрерывные дроби.
Л.А. Гутник **33**

Теоремы единственности мероморфных функций нескольких комплексных переменных.
Пей-Чу Ху, Чунг Чун Янг **45**

Нормальное семейство и распределенные многочлены мероморфных функций.
Фенг Лю, Жан Фенг Ксю **54**

Разряженные гипергеометрические системы.
Тимур Садыков **64**

Группы отражений ранга три и системы уравнений диформализации.
Джиро Секигучи **77**

Метод мультиполей для некоторых эллиптических краевых задач с разрывным коэффициентом.
С.Л. Скороходов, В.И. Власов **89**

Обобщенный потенциал двойного слоя для эллиптических систем второго порядка.
А.П. Солдатов **103**

Разделенное множество и нормальная функция голоморфных отображений.
Жан Фенг Ксю **110**

Единственность мероморфных функций с тремя разделенными значениями.
Янг-Пинг Ванг, Ксяо Линг Ванг,
Линг Ди Хуанг **118**

Системы уравнений первого порядка главного типа на плоскости.
Н.А. Жура **126**

Сведения об авторах. **132**

Информация для авторов. **133**

Оригинал-макет *Бекназаров М.Н.*
e-mail: beknazarov@bsu.edu.ru
Подписано в печать 25.12.2009 г.
Формат 60×84/8
Гарнитура Georgia, Impact
Усл. п. л. 15,57
Тираж 500 экз.
Заказ 248

Подписные индексы:
в каталоге агентства «Роспечать» – 81631,
в объединенном каталоге
«Пресса России» – 39723

Оригинал-макет тиражирован
в издательстве Белгородского государственного университета
Адрес: 308015, г. Белгород, ул. Победы, 85

№ 13 (68) 2009
Issue 17/1
SCIENTIFIC REVIEWING JOURNAL

Founded in 1995

Belgorod State University Scientific bulletin
Mathematics Physics

НАУЧНЫЕ ВЕДОМОСТИ Белгородского государственного университета
Математика Физика

Founder:
State educational establishment of higher professional education "Belgorod State University"

Publisher:
Belgorod State University
BSU Publishing house

The journal is registered in Federal service of control over law compliance in the sphere of mass media and protection of cultural heritage

Certificate of registration of mass media
ПИ № ФС 77-21121 May, 19 2008.

CONTENTS

Introduction. **5**

Residues of logarithmic differential forms.
A.G. Aleksandrov **6**

Logarithmic differential forms and analysis of complex dynamical systems.
A.G. Aleksandrov, A.A. Castro, Jr., V.A. Gruzman **20**

On some expansions of the number $\zeta(3)$ in continued fractions.
L.A. Gutnik **33**

Uniqueness theorems of meromorphic functions in several complex variables .
Pei-Chu Hu, Chung-Chun Yang **45**

Normal family and the shared polynomials of meromorphic Functions.
Feng Lu Jun-Feng Xu **54**

Sparse hypergeometric systems .
Timur Sadykov **64**

Reflection groups of rank three and systems of uniformization equations .
Jiro Sekiguchi **77**

The multipole method for certain elliptic equation with discontinuous coefficient .
S.L. Skorokhodov, V.I. Vlasov **89**

Generalized potentials of double layer for second order elliptic systems.
A.P. Soldatov **103**

Sharing set and normal function of holomorphic functions.
Jun-Feng Xu **110**

Uniqueness of meromorphic functions that share three values.
Jian-Ping Wang, Xiao-Ling Wang, Ling-Di Huang **118**

The first order system equations of a principal type on the plane.
N.A. Zhura **126**

Information about Authors. **132**

Information for Authors. **133**

Dummy layout by M.N Beknazarov
e-mail: beknazarov@bsu.edu.ru
Passed for printing 25.12.2009
Format 60×84/8
Typeface Georgia, Impact
Printer's sheets 15,57
Circulation 500 copies
Order 248

Subscription reference in Rospechat' agency catalogue – 81631,
In joint catalogue Pressa Rossii – 39723

Dummy layout is replicated at
Belgorod State University Publishing House
Address: 85, Pobedy str., Belgorod, Russia, 308015

ВВЕДЕНИЕ

25 октября 2009 г. в БелГУ состоялась вторая сессия Российско-Китайского симпозиума «Комплексный анализ и его приложения. Первая сессия прошла в Москве в Институте проблем управления РАН с 21 по 24 октября 2009 г. Эти сессии были организованы Российским фондом фундаментальных исследований (РФФИ) и Национальным фондом естественных наук Китая (НФЕК) в рамках совместного научного проекта «Комплексный анализ и его приложения» при поддержке ИПУ РАН и БелГУ.
В организационный комитет сессии вошли А. П. Солдатов (Белгород, Россия) председатель, Ч.-Ч. Янг (Гонконг, КНР), почетный председатель, А. Г. Александров (Москва, Россия), П.Ху (Шандонг, КНР).

Сессия была посвящена комплексному анализу и его приложениям в теории дифференциальных уравнений, динамических систем, в топологии и геометрии, в теории функций и пр. Целью симпозиума явилось обсуждение наиболее актуальных проблем комплексного анализа и его приложений, выявление новых перспектив развития научных исследований, а также возможностей для совместных научных исследований.

В состав иностранных участников конференции вошли такие крупные математики, как проф. П.Ху (Hu Peichu – КНР), проф. Ч.Ч. Янг (Chung Chun Yang - Гонконг), проф. Ванг (Wang Jian Ping – КНР), проф. Ксю (Jun Feng Xu - КНР), проф Киан Т. (Tao Qian – Австралия) , проф. Ш. Таджима (Sh. Tajima – Япония) и др. С обзорным докладом выступил действительный член польской АН, проф. Б. Боярский. В географии научных докладов представлены также Харьков и Донецк (Украина), Алма-Ата (Казахстан), Ереван (Армения).

В рамках научной программы с российской стороны приняли участие известные специалисты в области комплексного анализа и дифференциальных уравнений из многих научных центров страны, включая Москву, Санкт-Петербург, Новосибирск, Челябинск, Уфу, Красноярск и университеты центрального региона. В частности, от Московского государственного университета выступили с докладами профессора В.Н. Чубариков (декан механико-математического факультета) и Е.В. Радкевич, А.В. Боровских (кафедра дифференциальных уравнений), Вычислительный центр им. Дородницына РАН представлен проф. В.И. Власовым, приняли также участие проф. А.И. Назаров (Санкт-Петербургский госуниверситет) и проф. А.И. Кожанов (Новосибирский госуниверситет). В работе симпозиума широкое участие приняли ведущие математики Белгородского госуниверситета – проф. А.П. Солдатов, проф.А.М. Мейрманов, проф. О.М. Пенкин, проф. А.В. Глушак, проф. С.А. Гриценко и др. Труды симпозиума представлены в нескольких номерах журнала «Научные ведомости Белгородского государственного университета», включая настоящий выпуск.

УДК 515.17

RESIDUES OF LOGARITHMIC DIFFERENTIAL FORMS

A.G. Aleksandrov

Institute for Control Sciences, Russian Academy of Sciences,
Profsojuznaja str., 65, Moscow, 117997, Russian Federation, e-mail: ag_aleksandrov@mail.ru

Abstract. In this note we give an elementary introduction to the theory of logarithmic differential forms and their residues. In particular, we consider some properties of logarithmic differential forms related with properties of the torsion holomorphic differentials on singular hypersurfaces, briefly discuss the definitions of residues due to Poincaré, Leray and Saito, and then explain an elegant description of the modules of regular meromorphic differential forms in terms of residues of meromorhic differential forms logarithmic along a hypersurface with arbitrary singularities.

Keywords: logarithmic differential forms, residue-forms, residue map, regular meromorphic differential forms, torsion holomorphic differentials.

Introduction

From the historical point of view, the concept of logarithmic differential form had its origin in the classical theory of residues. The term "residue"(together with its formal definition) appeared for the first time in an article by A.Cauchy (1826), although one can find such a notion as implicit in Cauchy's prior work (1814) about the computation of particular integrals which were related with his research towards hydrodynamics. For the next three-four years, Cauchy developed residue calculus and applied it to the computation of integrals, the expansion of functions as series and infinite products, the analysis of differential equations, and so on ...

Though it was already transparent in the pioneer work of N.Abel, a major step towards the elaboration of the residue concept was made by H. Poincaré who introduced in 1887 the notion of differential residue 1-form attached to any rational differential 2-form in $\mathbf{C}^2$ with simple poles along a smooth complex curve. Subsequently É. Picard (1901), G. de Rham (1932/36), A.Weil (1947) obtained a series of similar results about residues of meromorphic forms of degree 1 and 2 on complex manifolds; such developments were associated with cohomological ideas, leading to the formulation of cohomological residue formulae. Such cohomological ideas were later pursued by G. de Rham (1954) and J.Leray (1959) who defined and studied residues of d-closed C^∞ regular differential q-forms on $S \setminus D$ with poles of the first order along a smooth hypersurface D in some complex manifold S, $q \geq 1$.

In 1972 J.-B. Poly [24] proved that Leray residue is well determined for any (not necessarily d-closed) *semi-meromorphic* differential forms ω as soon as ω and $d\omega$ have simple poles along a hypersurface.

In fact, for the first time these two conditions were considered by P.Deligne [11]; he introduced the notion of meromorphic differential forms with *logarithmic poles* along a divisor, normal crossings of smooth irreducible components. In such context this notion was extensively studied in algebraic geometry and in differential equations by many authors (for example, by Ph.Griffiths,

A.G. Alexandrov partially supported by the Russian Foundation of Basic Research (RFBR) and by the National Natural Science Foundation of China (NSFC) in the framework of the bilateral project "Complex Analysis and its applications"(project No. 08-01-92208)

J.Steenbrink, N.Katz). As a result in 1975, Kyoji Saito [25] considered *meromorphic* differential forms satisfied these conditions in the case of divisors with *arbitrary* singularities. Somewhat later, his note has been published in a volume [26] of the RIMS-publication series, which is not accessible to many of those interested in the subject. Saito established the basic properties of logarithmic differential forms and studied some applications to computing Gauss-Manin connection associated with the minimal versal deformations of simple hypersurface singularities of types A_2 and A_3. In 1980 a paper by Saito [27] was published; it contains an essential part of materials of the above mentioned works. Among other things, in this paper a general notion and important properties of residues of logarithmic differential forms are discussed in detail.

At present time the theory of logarithmic differential forms is exploited fruitfully in various fields of modern mathematics. Among them, one can mention the following:

complex algebraic geometry (the cohomology theory of algebraic varieties and Hodge theory [12], [10], [29], etc.),

topology and geometry (the theory of arrangements of real and complex hyperplanes [21], [7], the fundamental group of the complement of a singular hypersurface [19], etc.),

the theory of singularities, the deformation theory and the theory of Gauss-Manin connexion [26], [4], etc.,

the theory of $\mathcal{D}$-modules, the microlocal analysis, the theory of differential equations [11], [22], the theory of flat coordinate systems [28], etc.,

complex analysis (the theory of Abel's integrals [15], Torelli theorems, the theory of primitive forms and their periods [16], etc.),

the theory of special functions (generalized hypergeometric functions [12], etc.),

mathematical and theoretical physics (the theory of Frobenius varieties and the topological field theory [20], etc.)

Of course, this list is quite incomplete and can be easily extended by the specialists in related fields of mathematics.

Following our previous work [3] in this note we give an elementary introduction to the theory of logarithmic differential forms and their residues. In Section 1 we recall the basic notations, definitions and properties of logarithmic differential forms along a reduced hypersurface in a complex analytic manifold. In Section 2 we consider some relations of logarithmic differential forms and torsion holomorphic differentials on singular hypersurfaces. In the next sections we briefly discuss the definitions of Poincaré, de Rham, Leray and Saito residues, and apply the theory of regular meromorphic differential forms to the case of singular hypersurfaces. Among other things, we obtain a highly elegant description of these modules on an arbitrary singular hypersurface D in terms of residues of logarithmic differential forms.

1 Logarithmic differential forms

Let U be an open subset of $\mathbf{C}^m$, and let D be a hypersurface defined by an equation $h(z) = 0$, where $h(z) = h(z_1, \dots, z_m)$ is a holomorphic function in U, and $z_1, \dots, z_m$ is a system of coordinates. Suppose that D is *reduced,* that is, $h(z)$ has no multiple factors.

Definition 1.1 ([25], [27]) A meromorphic differential q-form ω, $q \geq 0$, on U is called *logarithmic* (along a divisor D) if ω and its differential $d\omega$ have poles along D at worst of the first order. It means that $h\omega$ and $hd\omega$ are *holomorphic* differential forms on U.

Remark 1.2 In fact, for the first time the above two conditions appeared in a work by Deligne (see [11], Prop. 3.2, (i), p.72) who studied meromorphic differential forms with *logarithmic poles* along divisors with normal crossings (thus, such a divisor D is the union of its smooth irreducible components).

In practical computations, the second condition is usually replaced by the condition "$dh \wedge \omega$ is a *holomorphic* differential form on U"; both conditions are equivalent, in view of the identity $d(h\omega) = dh \wedge \omega + h \cdot d\omega$.

Let S be an m-dimensional complex manifold, and let $\Omega^{\bullet}_S = \big(\Omega^q_S, d\big)_{q=0,1,\dots}$ be the de Rham complex of germs of holomorphic differential forms on S, whose terms, locally at the point $x \in S$, are defined as follows:

$$\Omega^q_{S,x} = \mathcal{O}_{S,x}\langle \dots, dz_{i_1} \wedge \dots \wedge dz_{i_q}, \dots \rangle, \quad (i_1, \dots, i_q) \in [1, m].$$

Let D be a reduced hypersurface of S, and let $h = 0$ be an equation of D, locally at the point $x \in D$. A meromorphic q-form ω is logarithmic along D at x, if $h\omega$ and $hd\omega$ are holomorphic. We denote the $\mathcal{O}_{S,x}$-module of germs of logarithmic q-form at x and the corresponding *sheaf* of logarithmic differential q-form on S by $\Omega^q_{S,x}(\log D)$ and $\Omega^q_S(\log D)$, respectively. Thus, the $\mathcal{O}_S$-module $\Omega^q_S(\log D)$ is a submodule of $\Omega^q_S(\star D)$, consisting of all the "differential forms with polar singularities along D." Obviously, the sheaves $\Omega^q_S(\log D)$ and Ω^q_S coincide off the divisor D, for all $q \geq 0$. By definition,

$$\Omega^0_{S,x}(\log D) \cong \Omega^0_{S,x} \cong \mathcal{O}_{S,x}, \quad \Omega^m_{S,x}(\log D) \cong \frac{1}{h}\Omega^m_{S,x}.$$

In what follows, when we consider the local situation the point x will be taken to be 0 for simplicity. We shall also assume that U is an open subset of $\mathbf{C}^m$ containing the origin.

Example 1.3 Suppose $D \subset U$ be a hyperplane or, more generally, a smooth hypersurface defined by the equation $z_1 = 0$. Then

$$\Omega^1_{S,0}(\log D) \cong \mathcal{O}_{S,0}\left\langle \frac{dz_1}{z_1}, dz_2, \dots, dz_m \right\rangle$$

is a *free* $\mathcal{O}_{S,0}$-module of rank m, generated by the forms $dz_1/z_1, dz_2, \dots, dz_m$. Moreover,

$$\Omega^q_{S,0}(\log D) \cong \bigwedge^q \Omega^1_{S,0}(\log D), \quad 1 \leq q \leq m.$$

Example 1.4 More generally, let us consider the case when D is the union of $k \leq m$ coordinates hyperplanes in $S = \mathbf{C}^m$. In other words, D is a *strong normal crossing.* This case considered in many works published before Saito's preprint [25]. Then the defining equation of D is written as follows: $h = z_1 \cdots z_k = 0$, and an easy calculation shows that

$$\Omega^1_{S,0}(\log D) \cong \mathcal{O}_{S,0}\left\langle \frac{dz_1}{z_1}, \dots, \frac{dz_k}{z_k}, dz_{k+1}, \dots, dz_m \right\rangle,$$

and for all $1 \leq q \leq m$ there are the following isomorphisms

$$\Omega^q_{S,0}(\log D) \cong \bigwedge^q \Omega^1_{S,0}(\log D).$$

Thus, $\Omega^q_{S,0}(\log D)$ is a *free* $\mathcal{O}_{S,0}$-module of rank $\binom{m}{q}$.

The following statement is a direct consequence of the basic definition (see [1], or [2], §1).

Claim 1.5 *Let $D \subset U$ be a reduced hypersurface defined by the equation $h = 0$. Then for any $q \geq 1$ there exists a natural isomorphism of $\mathcal{O}_{S,0}$-modules*

$$\Omega^q_{S,0} \bigcap \left((dh/h) \wedge \Omega^{q-1}_{S,0}\right) \cong dh \wedge \Omega^{q-1}_{S,0}(\log D).$$

Proof. Let us remark at first that there is a natural inclusion

$$\Omega^q_{S,0} \bigcap \left((dh/h) \wedge \Omega^{q-1}_{S,0}\right) \hookrightarrow dh \wedge \Omega^{q-1}_{S,0}(\log D).$$

If an element $\omega \in \Omega^q_{S,0}$ belongs to the $\mathcal{O}_{S,0}$-module on the left side, then it can be represented in the form $\omega = (dh/h) \wedge \eta$ for some $\eta \in \Omega^{q-1}_{S,0}$. Hence, by definition,

$$(\eta/h) \in \Omega^{q-1}_{S,0}(\log D) \Rightarrow \omega \in dh \wedge \Omega^{q-1}_{S,0}(\log D),$$

and we obtain the desirable inclusion. On the other hand, $h \cdot \Omega^{q-1}_{S,0}(\log D) \subseteq \Omega^{q-1}_{S,0}$. Multiplication by $\wedge dh$ induces the map

$$dh \wedge \Omega^{q-1}_{S,0}(\log D) \longrightarrow \frac{dh}{h} \wedge \Omega^{q-1}_{S,0}.$$

Obviously this gives us the inverse map to the first inclusion. This completes the proof of Claim.

Lemma 1.6 ([27], $(1.1), iii))$**)** *Let ω be a meromorphic q-form on U, $q \geq 0$, and let $D \subset U$ be a hypersurface as above. Then ω is logarithmic along D if and only if there exist a holomorphic function g defining a hypersurface $V \subset U$, a holomorphic $(q-1)$-form ξ and a holomorphic q-form η on U such that*

a) $\dim_{\mathbf{C}} D \cap V \leq m - 2$,

b) $g\omega = \dfrac{dh}{h} \wedge \xi + \eta.$

Proof. For simplicity let us consider the case $m = 2$. Suppose that ω is a logarithmic q-form. Then we have

$$\omega = \frac{a_1 dz_1 + a_2 dz_2}{h}, \quad dh \wedge \omega = \frac{h'_1 a_2 - h'_2 a_1}{h} dz_1 \wedge dz_2 \stackrel{\text{def}}{=} b(z) dz_1 \wedge dz_2,$$

where a_1, a_2 and $b(z)$ are holomorphic, and $h'_i = \partial h / \partial z_i,\ i = 1, 2$. Further,

$$h'_1 \omega = \frac{h'_1 a_1 dz_1 + h'_1 a_2 dz_2}{h} =$$

$$\frac{h'_1 a_1 dz_1 + h'_2 a_1 dz_2}{h} + \frac{h'_1 a_2 - h'_2 a_1}{h} dz_2 = \frac{dh}{h} \wedge a_1 + b(z) dz_2.$$

It means that

$$h'_1 \omega = \frac{dh}{h} \wedge a_1 + b(z) dz_2.$$

There is analogous representation for $h'_2\omega$, and hence for any $g\omega$, where $g \in J(h) = (h'_1, h'_2)$, the Jacobian ideal of h. Since D is reduced, there is a function $g \in J(h)$ defining a non-zero divisor in $\mathcal{O}_D/(h)$ as required in the condition a).

Conversely, the relation b) implies that

$$h\omega = dh \wedge \frac{\xi}{g} + \frac{\eta}{g},$$

that is, $h\omega$ and $dh \wedge \omega$ are holomorphic in codimension ≥ 2, hence, in virtue of the Riemann extension theorem, they are holomorphic everywhere. This completes the proof when $m = 2$. The general case can be considered analogously.

Corollary 1.7 ([25]) *With the preceding notations the following conditions are equivalent:*

1) $\omega \in \Omega^q_S(\log D)$,
2) $\dfrac{\partial h}{\partial z_i}\omega \in \dfrac{dh}{h} \wedge \Omega^{q-1}_U + \Omega^q_U$ *for all* $i = 1, \dots, m$.

Corollary 1.8 *The sheaves* $\Omega^q_S(\log D)$, $q = 0, 1, \dots, m$, *are* $\mathcal{O}_S$*-modules of finite type; the direct sum* $\oplus^m_{q=0}\Omega^q_S(\log D)$ *is an* $\mathcal{O}_S$*-exterior algebra closed under the exterior differentiation* d.

As a consequence, $\Omega^q_S(\log D)$ are *coherent* sheaves of $\mathcal{O}_S$-modules for all $q \geq 0$.

2 Torsion differentials

In this section we consider simple relations between logarithmic differential forms and torsion holomorphic differentials on hypersurfaces with singularities. By definition, $\mathcal{O}_{D,0} = \mathcal{O}_{S,0}/(h)\mathcal{O}_{S,0}$, and

$$\Omega^q_{D,0} = \Omega^q_{S,0}/\big(h \cdot \Omega^q_{S,0} + dh \wedge \Omega^{q-1}_{S,0}\big), \quad q \geq 1.$$

Thus, $\Omega^q_{D,0}$ is the $\mathcal{O}_{D,0}$-module of germs of *holomorphic* differential forms on the *hypersurface* D at the point $0 \in D$. The module $\Omega^1_{D,0}$ is usually called the module of Kähler regular differentials. The standard differentiation d induces the action on $\Omega^q_{D,0}$ denoted by the same symbol. Thus, the de Rham complex of sheaves of germs of holomorphic differential forms on D is well defined:

$$\Omega^\bullet_D = \big(\Omega^q_D, d\big)_{q=0,1,\dots}.$$

For completeness, recall the notion of torsion. Given a commutative ring A with the total ring of fractions F, and an A-module N of finite type, we consider the kernel of the canonical map $\iota\colon N \to N \otimes_A F$, the *torsion* submodule of N, and denote it by $\operatorname{Tors} N$; it consists of all the elements of N which are killed by non-zero divisors of A.

It is well-known that torsion differentials $\operatorname{Tors}\Omega^q_{D,0}$ play a key role in analysis of topology and geometry of singular varieties. In the case of an isolated n-dimensional singularity $(D, 0)$, the torsion modules $\operatorname{Tors}\Omega^q_{D,0}$ are trivial for all $q = 1, \dots, n-1$, while $\operatorname{Tors}\Omega^n_{D,0}$ is a finite dimensional vector space. Furthermore, if D is the quasi-homogeneous germ of a hypersurface or complete intersection with isolated singularities then $\dim_{\mathbf{C}}\operatorname{Tors}\Omega^n_{D,0} = \mu$, where μ is the Milnor number of D; it is a very important topological invariant of the singularity.

The following examples show that generators of the module of logarithmic differential forms are naturally expressed through torsion differentials on D.

Example 2.1 Suppose $S = \mathbf{C}^2$ and consider the hypersurface D given by the equation $h = xy = 0$. It is a plane curve with a node. In other words, it is an A_1-singularity, a very particular case of strong normal crossing from Example 1.4. Then

$$\Omega^1_{S,0}(\log D) \cong \mathcal{O}_{S,0}\left\langle \frac{dx}{x}, \frac{dy}{y} \right\rangle, \ \Omega^2_{S,0}(\log D) \cong \mathcal{O}_{S,0}\left\langle \frac{dx \wedge dy}{xy} \right\rangle$$

are *free* $\mathcal{O}_{S,0}$-modules of rank 2 and 1, respectively. In this case there is also the following representation: $\Omega^1_{S,0}(\log D) \cong \mathcal{O}_{S,0}\langle dh/h,\ \theta/h\rangle$, where $\theta = ydx - xdy$. It is not difficult to verify that $\theta \in \operatorname{Tors}\Omega^1_{D,0}$. Indeed, taking a non-zero divisor $(x+y) \in \mathcal{O}_{D,0}$ one obtains the following identities in $\Omega^1_{D,0}$:

$$(x+y)\cdot\theta = xydx - x^2dy + y^2dx - xydy = -(x-y)dh + 2h(dx - dy) \equiv 0.$$

Moreover, in this case, $\operatorname{Tors}\Omega^1_{D,0} \cong \mathcal{O}_{D,0}\langle\theta\rangle \cong \mathbf{C}\langle\theta\rangle,\ \mu = 1$.

Example 2.2 (cf. [30]) With the preceding notations let $D \subset S$ be a plane curve with a cusp given by the equation $h = x^2 - y^3 = 0$. In other words, it is an A_2-singularity. Easy calculations show that

$$\Omega^1_{S,0}(\log D) \cong \mathcal{O}_{S,0}\left\langle \frac{dh}{h}, \frac{2ydx - 3xdy}{h} \right\rangle, \ \Omega^2_{S,0}(\log D) \cong \mathcal{O}_{S,0}\left\langle \frac{dx \wedge dy}{h} \right\rangle$$

are again free $\mathcal{O}_{S,0}$-modules of rank 2 and 1, respectively. Notice that the numerator of the second generator of $\Omega^1_{S,0}(\log D)$, the differential 1-form $\theta = 2ydx - 3xdy$, represents an element of the *torsion* submodule $\operatorname{Tors}\Omega^1_{D,0} \subset \Omega^1_{D,0}$. Indeed, in our case $A = \mathcal{O}_{D,0} \cong \mathbf{C}\langle t^2, t^3\rangle,\ N = \Omega^1_{D,0},\ F = \mathbf{C}(t)$, and the mapping ι is given by the normalization of D, that is, $x = t^3,\ y = t^2$. Thus, $\iota(\theta) = \iota(2ydx - 3xdy) = 2t^2dt^3 - 3t^3dt^2 = 0$, that is, $\theta \in \operatorname{Ker}(\iota) \cong \operatorname{Tors}\Omega^1_{D,0}$. Equivalently, take a non-zero divisor $x \in \mathcal{O}_{D,0}$. One then obtains $x\cdot\theta = 2xydx - 3x^2dy = 5hdx - 3xdh \equiv 0$ in $\Omega^1_{D,0} = \Omega^1_{S,0}/(h\cdot\Omega^1_{S,0} + dh\wedge\mathcal{O}_{S,0})$. Further calculations show (cf. [30]) that $\operatorname{Tors}\Omega^1_{D,0} \cong \mathcal{O}_{D,0}\langle\theta\rangle \cong \mathbf{C}\langle\theta,\ y\cdot\theta\rangle$, that is, $\mu = 2$.

Proposition 2.3 ([1]) *For $q = 1, \ldots, m$, there are exact sequences of $\mathcal{O}_{S,0}$-modules*

$$0 \longrightarrow \Omega^{q-1}_{S,0}/h\cdot\Omega^{q-1}_{S,0}(\log D) \xrightarrow{\wedge dh} \Omega^q_{S,0}/h\cdot\Omega^q_{S,0} \longrightarrow \Omega^q_{D,0} \longrightarrow 0,$$

$$0 \longrightarrow \Omega^q_{S,0}/dh\wedge\Omega^{q-1}_{S,0}(\log D) \xrightarrow{\cdot h} \Omega^q_{S,0}/dh\wedge\Omega^{q-1}_{S,0} \longrightarrow \Omega^q_{D,0} \longrightarrow 0,$$

$$0 \longrightarrow \Omega^q_{S,0} + \frac{dh}{h}\wedge\Omega^{q-1}_{S,0} \longrightarrow \Omega^q_{S,0}(\log D) \xrightarrow{\cdot h} \operatorname{Tors}\Omega^q_{D,0} \longrightarrow 0,$$

where the homomorphisms of exterior and usual multiplication are denoted by $\wedge dh$ and by $\cdot h$, respectively.

Proof. The exactness of the first and second sequences follows directly from the basic Definition 1.1. Let us consider a differential q-form $\omega \in \Omega^{q-1}_{S,0}$ represented an element of the quotient $\Omega^{q-1}_{S,0}/h\cdot\Omega^{q-1}_{S,0}(\log D)$. Suppose $dh\wedge\omega = h\cdot\eta,\ \eta \in \Omega^q_{S,0}$, and set $\tilde\omega = \omega/h$. It is obvious that $h\tilde\omega$ and $dh\wedge\tilde\omega$ are holomorphic, hence $\tilde\omega \in \Omega^{q-1}_{S,0}(\log D)$ by definition. Thus the first sequence is exact from the left. Evidently it is exact from the right too. In the same way, one can easily prove the exactness of the second sequence. The exactness from the left of the third sequence follows from definition. In view of Lemma 1.6, it is clear that $\mathrm{Im}\,(\cdot h) \subseteq \mathrm{Tors}\,\Omega^q_{D,0}$ because for a non-zero divisor g one has the following chain of implications:

$$g\omega = \frac{dh}{h} \wedge \xi + \eta \quad \Rightarrow g(h\omega) = dh \wedge \xi + h\eta \equiv 0 \quad \Rightarrow h\omega \in \mathrm{Tors}\,\Omega^q_{D,0}.$$

Now let take an element $\omega \in \mathrm{Tors}\,\Omega^q_{D,0}$. By definition, there is a non-zero divisor $g \in \mathcal{O}_{D,0}$ such that $g\,\omega = 0$. We will denote by g and ω their representatives in $\mathcal{O}_{S,0}$ and $\Omega^q_{S,0}$, respectively. Then one has $g{\cdot}\omega = dh \wedge \xi + h{\cdot}\eta,\ \xi \in \Omega^{q-1}_{S,0},\ \eta \in \Omega^q_{S,0}$. Since g is a non-zero divisor, the condition $b)$ of Lemma 1.6 is satisfied. This implies that $\omega/h = \tilde{\omega} \in \Omega^q_{S,0}(\log D)$, that is, $\omega \in \mathrm{Im}\,(\cdot h)$.

Remark 2.4 It is well-known [14] that $\mathrm{Tors}\,\Omega^q_{D,0} = 0,\ 0 < q < c$, where $c = \mathrm{codim}\,(\mathrm{Sing}\,D, D)$ and $\mathrm{Sing}\,D$ is the singular locus of D. On the other side, any reduced hypersurface (or complete intersection) D is normal if and only if $c \geq 2$ by virtue of Serre's criterion ("R_1 and S_2 conditions imply normality"). Hence, when D is *normal* then the exact sequence of Proposition 2.3 implies the following isomorphisms

$$\Omega^q_{S,0}(\log D) \cong \Omega^q_{S,0} + \frac{dh}{h} \wedge \Omega^{q-1}_{S,0}, \quad 1 \leq q < c.$$

It is not difficult to see that the support of $\mathrm{Tors}\,\Omega^1_D$ is contained in the singular locus $\mathrm{Sing}\,D$ of the hypersurface D. Moreover, there is a system of generators of $\mathcal{O}_D$-module $\mathrm{Tors}\,\Omega^1_D$ containing at least $m-1$ elements.

Corollary 2.5 *There are the following long exact sequences of $\mathcal{O}_{S,0}$-modules*

$$0 \to \Omega^q_{S,0} + \tfrac{dh}{h} \wedge \Omega^{q-1}_{S,0} \to \Omega^q_{S,0}(\log D) \xrightarrow{\cdot h} \Omega^q_{D,0} \to \Omega^q_{D,0}/\mathrm{Tors}\,\Omega^q_{D,0} \to 0,$$
$$0 \to dh \wedge \Omega^{q-1}_{S,0}(\log D) \to \Omega^q_{S,0} \oplus \tfrac{dh}{h} \wedge \Omega^{q-1}_{S,0} \to \Omega^q_{S,0}(\log D) \xrightarrow{\cdot h} \mathrm{Tors}\,\Omega^q_{D,0} \to 0.$$

Proof. This is an immediate consequence of Proposition 2.3 and Claim 1.5.

Remark 2.6 The last sequence is very useful in computing the torsion modules in the case when $\Omega^q_{S,0}(\log D)$ is a *free* $\mathcal{O}_{S,0}$-module; it gives us an $\mathcal{O}_{S,0}$-free *resolution* of the torsion module. Following P.Cartier [9] a hypersurface $D \subset S$ is called *Saito divisor* or, more often, *Saito free* divisor if for some $q \geq 1$ and, consequently, for all q, the $\mathcal{O}_S$-module $\Omega^q_S(\log D)$ is locally free. For example, the discriminants of the minimal versal deformations of isolated hypersurface or complete intersection singularities are Saito free divisors.

3 Poincaré residue

The following construction [15] is a direct generalization of the original Poinacaré definition of the residue 1-form associated with any rational 2-form in $\mathbf{C}^2$.

Let ω be a meromorphic differential m-form on an m-dimensional complex analytic *manifold* S with a polar divisor $D \subset S$. Thus, locally we have a representation:

$$\omega = \frac{f(z)dz_1 \wedge \ldots \wedge dz_m}{h(z)},$$

where f and h are germs of holomorphic functions, and h is a local equation of D. By definition, the Poincaré residue $\mathrm{rés}_D(\omega)$ is a meromorphic $(m-1)$-form on D whose singularities are contained in the singular locus $\mathrm{Sing}\, D \subset D$. To define this form explicitly, let us note that at each point $x \in D \setminus \mathrm{Sing}\, D$ at least one of the derivatives of h does not vanish:

$$\left.\frac{\partial h}{\partial z_i}\right|_{z=x} \neq 0.$$

Then the Poincaré residue of ω in a small neighbourhood of x is defined as follows:

$$\mathrm{rés}_D(\omega) = (-1)^{m-i} \left.\frac{f(z) dz_1 \wedge \ldots \wedge \widehat{dz_i} \wedge \ldots \wedge dz_m}{\partial h(z)/\partial z_i}\right|_D .$$

It is not difficult to verify that this restriction depends neither on the index i nor on the local coordinates and on defining equations of D. Moreover, the Poincaré residue is holomorphic on the complement $S \setminus D$. When D is *smooth,* one can take $h(z) = z_m$, and then, as usually,

$$\mathrm{rés}_D\left(\frac{f(z) dz_1 \wedge \ldots \wedge dz_m}{z_m}\right) = f(z) dz_1 \wedge \ldots \wedge dz_{m-1},$$

that is, $\mathrm{rés}_D(\omega)$ is holomorphic on D. As a result one has the following sequence of sheaves

$$0 \longrightarrow \Omega_S^m \longrightarrow \Omega_S^m(D) \xrightarrow{\mathrm{rés}} \Omega_D^{m-1} \longrightarrow 0 \;,$$

where $\Omega_S^m(D)$ denotes the sheaf of meromorphic forms on S having a simple pole along the divisor D. In particular, one concludes that the germ of every holomorphic $(m-1)$-form on the nonsingular divisor D is a Poincaré residue. It is obvious that this is true globally when the first cohomology group vanishes: $H^1(S, \Omega_S^m) = 0$.

4 Leray residue-form

As remarked in Introduction De Rham and Leray considered d-closed C^∞ regular differential forms on $S \setminus D$ having simple poles on D, where D is a submanifold of codimension 1 in a smooth manifold S. In particular, they proved that locally for such a form there is the following represenation:

$$(*) \qquad \omega = \frac{dh}{h} \wedge \xi + \eta,$$

where ξ and η are germs of regular differential forms on S. In fact, $\xi_{|D}$ is globally and uniquely determined; it is closed on D. If ω is holomorphic on $S \setminus D$ then the form $\xi_{|D}$ is holomorphic on D. The form $\xi_{|D}$ is called the Leray *residue-form* on D; it is denoted by $\mathrm{res}[\omega]$. It is not difficult to see that the definition of the Leray residue-form generalizes the Ponacaré residue described above.

Similarly to the construction from the end of the previous section, making use of local representation $(*)$, for any $q = 1, \ldots, m$ one gets (see [23]) the exact sequence

$$0 \longrightarrow \Omega_S^q \longrightarrow \Omega_S^q(D) \xrightarrow{\mathrm{res}} \Omega_D^{q-1} \longrightarrow 0,$$

which is equivalent, since the divisor D is a *smooth* hypersurface, to the sequence

$$0 \longrightarrow \Omega^q_S \longrightarrow \Omega^q_S(\log D) \xrightarrow{\text{res}} \Omega^{q-1}_D \longrightarrow 0\ .$$

Below we show that a generalization of this sequence to the case when the divisor D has arbitrary singularities requires more delicate considerations.

5 Saito residue map

In fact, Leray considered d-closed forms on $S\setminus D$ in order to construct a natural homomorphism of cohomology spaces $H^p(S\setminus D) \to H^{p-1}(D)$, and then the co-boundary homomorphisms of homology groups $H_{p-1}(D) \to H_p(S\setminus D)$, the main ingredient of his famous residue-formula.

Furthermore, in 1972 J.-B. Poly [24] proved that the representation $(*)$ are valid for any *semi-meromorphic* differential form ω as soon as ω and $d\omega$ have simple poles along a smooth hypersurface $D \subset S$. By definition, a differential form ω is called semi-meromorphic when locally all its coefficients can be represented as quotient of smooth and holomorphic functions. Hence, the Leray residue is also well determined for such forms without assumption on their d-closedness.

Following Saito [27] we describe a natural generalization of the Leray residue for *meromorphic* differential forms satisfying the above two conditions for a divisor D with arbitrary singularities, that is, for *logarithmic* differential forms in the sense of Definition 1.1.

Let $D \subset S$ be a hypersurface, and let the sheaf $\mathcal{M}_D$ be the $\mathcal{O}_D$-module of germs of meromorphic functions on D.

Definition 5.1 (see [27], (2.2)**)** The (logarithmic) residue morphism is a homomorphism of $\mathcal{O}_S$-modules

$$\text{res.}: \ \Omega^q_S(\log D) \longrightarrow \mathcal{M}_D \otimes_{\mathcal{O}_D} \Omega^{q-1}_D,$$

defined locally as follows: taking the representation of the basic Lemma 1.6, for any $\omega \in \Omega^q_{S,0}(\log D)$ we set

$$\text{res.}\,\omega = \frac{1}{g}\cdot\xi.$$

Thus, the residue res. ω is the germ of the meromorphic $(q-1)$-form in the module $\mathcal{M}_{D,0}\otimes_{\mathcal{O}_{D,0}} \Omega^{q-1}_{D,0}$.

Claim 5.2 ([27], (2.5)**)** *Let $D \subset S$ be a hypersurface. Then for any $q \geq 1$ there exists the following exact sequence of $\mathcal{O}_S$-modules*

$$0 \longrightarrow \Omega^q_S \longrightarrow \Omega^q_S(\log D) \xrightarrow{\text{res.}} \mathcal{M}_D \otimes_{\mathcal{O}_D} \Omega^{q-1}_D.$$

Proof. Making use of the representation of logarithmic forms as in the definition of the symbol res. above, one obtains

$$\text{res.}\,\omega = 0 \Leftrightarrow g\omega \in \Omega^q_{S,0} \Leftrightarrow \omega \in \Omega^q_{S,0}.$$

This completes the proof.

Remark 5.3 In particular, for $q=1$ one has

$$0 \longrightarrow \Omega^1_S \longrightarrow \Omega^1_S(\log D) \xrightarrow{\text{res.}} \mathcal{M}_D \cong \mathcal{M}_{\widetilde{D}},$$

where $\widetilde{D}$ is the normalization of D. Moreover (see [27], Lemma (2.8)), if $\pi\colon \widetilde{D} \longrightarrow D$ is the morphism of normalization, then the image Im (res.) contains $\pi_*(\mathcal{O}_{\widetilde{D}})$ consisting of the so-called *weakly holomorphic* function on D, that is, of meromorphic functions, whose preimage becomes holomorphic on the normalization.

Remark 5.4 By this way we can consider the image of the logarithmic residue res. $\Omega^q_S(\log D)$ as an $\mathcal{O}_D$-module. Indeed, the definition of logarithmic forms implies that $h\cdot\bigl(\Omega^q_{S,0}(\log D)/\Omega^q_{S,0}\bigr) = 0$. Hence, the multiplication by $h\cdot$ annihilates Im (res.).

6 Regular meromorphic forms and Saito residue map

We are going to describe the image of the Saito residue map in terms of regular meromorphic forms for logarithmic differential forms with poles along a divisor $D\subset S$ with arbitrary singularities together with a generalization of the exact sequences from Section 3 and Section 4.

Now we consider the sheaves of $\mathcal{O}_D$-modules $\omega^q_D,\ q\geq 0$, called *regular meromorphic* differential q-forms on the hypersurface D. So let X, $\dim X = n \geq 1$, be the germ of an analytic subspace of an m-dimensional complex manifold S, and let $\omega^n_X = \operatorname{Ext}^{m-n}_{\mathcal{O}_S}(\mathcal{O}_X,\ \Omega^m_S)$ be the Grothendieck *dualizing module* of X.

Definition 6.1 ([18], [8]) The sheaves $\omega^q_X,\ q=0,1,\dots,n$, of *regular meromorphic* differential q-forms on X are defined as follows: ω^q_X consists of all meromorphic differential forms of order q on X such that $\omega\wedge\eta\in\omega^n_X$ for any $\eta\in\Omega^{n-q}_X$ or, equivalently, $\omega^q_X \cong \operatorname{Hom}_{\mathcal{O}_X}\bigl(\Omega^{n-q}_X,\ \omega^n_X\bigr)$.

Let us apply this Definition in the particular case when $X=D$ is a hypersurface, that is, $n=m-1$.

Claim 6.2 *Let $D\subset U$ be a reduced hypersurface. Then* res. $\Omega^{q+1}_S(\log D)\subseteq\omega^q_D$ *for all* $q = 0,1,\dots,m-1$.

Proof. Set $dz = dz_1\wedge\ldots\wedge dz_m$. Then with preceding notations one has a natural isomorphism $\omega^n_D \cong \mathcal{O}_D(dz/dh)$. That is, $\omega^q_D \cong \operatorname{Hom}_{\mathcal{O}_D}(\Omega^{n-q}_D,\ \mathcal{O}_D\bigl(dz/dh)\bigr)$ for all $q=0,1,\dots,n$. Then Corollary 1.7 implies that $\frac{\partial h}{\partial z_i}\cdot$res. $\Omega^q_S(\log D)\big|_U\subset\Omega^{q-1}_D\big|_{D\cap U}$ for all $i=1,\dots,m$, or, equivalently, $dh\wedge$ res. $\Omega^q_S(\log D)\big|_U\subset\Omega^q_D\big|_{D\cap U}$. This completes the proof.

Below we use an equivalent description of the regular meromorphic differential forms $\omega^q_D,\ q\geq 0$, on the hypersurface D obtained by D.Barlet in a more general context (see [8], Lemma 4). In fact, there is the following exact sequence of $\mathcal{O}_{D,0}$-modules:

$$0\longrightarrow\omega^q_{D,0}\xrightarrow{\mathcal{C}}\operatorname{Ext}^1_{\mathcal{O}_{S,0}}(\mathcal{O}_{D,0},\ \Omega^{q+1}_{S,0})\xrightarrow{\wedge dh}\operatorname{Ext}^1_{\mathcal{O}_{S,0}}(\mathcal{O}_{D,0},\ \Omega^{q+2}_{S,0}),\quad q\geq 0,$$

where $\omega^q_{D,0}\subset j_*j^*\Omega^q_{D,0}$ and $\mathcal{C}$ is induced by the multiplication by the fundamental class of D in S. Thus, $\mathcal{C}(v)$ corresponds to the Čech cocycle w/h such that $w=v\wedge dh$.

Theorem 6.3 ([2], §4) *Let $D \subset S$ be a reduced hypersurface. Then for any $q \geq 1$ there is the following exact sequence*

$$0 \to \Omega_S^{q+1} \longrightarrow \Omega_S^{q+1}(\log D) \xrightarrow{\text{res.}} \omega_D^q \to 0.$$

In particular, ω_D^q and $\operatorname{res.}\Omega_S^{q+1}(\log D)$ are isomorphic $\mathcal{O}_D$-modules.

Proof. It is sufficient to verify the statement locally. In view of Claim 6.2 it remains to prove that any element of ω_D^q can be represented as the residue of a logarithmic q-form.

Let $\mathcal{K}.(h)$ be the ordinary Koszul complex associated with h, that is,

$$0 \to \mathcal{O}_{S,0}e_0 \xrightarrow{d_0} \mathcal{O}_{S,0} \xrightarrow{d_{-1}} \mathcal{O}_{D,0} \to 0,$$

where $\mathcal{K}_1(h) = \mathcal{O}_{S,0}e_0,\ \mathcal{K}_0(h) = \mathcal{O}_{S,0}$ and $d_0(e_0) = h,\ \ d_{-1}(1) = 1$. Then we have the following piece of the dual exact sequence

$$\cdots \longrightarrow \operatorname{Hom}_{\mathcal{O}_{S,0}}(\mathcal{K}_0(h),\ \Omega_{S,0}^{q+1}) \xrightarrow{d^0} \operatorname{Hom}_{\mathcal{O}_{S,0}}(\mathcal{K}_1(h),\ \Omega_{S,0}^{q+1}) \longrightarrow$$
$$\longrightarrow \operatorname{Ext}^1_{\mathcal{O}_{S,0}}(\mathcal{O}_{D,0},\ \Omega_{S,0}^{q+1}) \to 0.$$

Hence, any element of $\operatorname{Ext}^1_{\mathcal{O}_{S,0}}(\mathcal{O}_{D,0},\ \Omega_{S,0}^{q+1})$ can be represented as a Čech 0-cochain (more explicitly, a 0-cocycle) in the following way

$$\nu/h \in \operatorname{Hom}_{\mathcal{O}_{S,0}}(\mathcal{K}_1(h),\ \Omega_{S,0}^{q+1}) \cong C^0_{(1)}(\Omega_{S,0}^{q+1}),$$

where $\nu \in \Omega_{S,0}^{q+1}$. Choose now an element $\nu \in \Omega_{S,0}^{q+1}$ such that

$$\frac{\nu}{h} \wedge dh \in \operatorname{Ext}^1_{\mathcal{O}_{S,0}}(\mathcal{O}_{D,0}, \Omega_{S,0}^{q+2}),$$

corresponds to the trivial element. That is, $\nu \wedge dh/h$ is defined by an element of $d^0\big(\operatorname{Hom}_{\mathcal{O}_{S,0}}(\mathcal{K}_0(h), \Omega_{S,0}^{q+2})\big)$. This means that $\nu \wedge dh = h \cdot \eta$ for some form $\eta \in \Omega_{S,0}^{q+2}$. The first exact sequence of Proposition 2.3 implies that $\nu \in h \cdot \Omega_{S,0}^{q+1}(\log D)$. Set $\tilde{\nu} = \mathcal{C}^{-1}(\nu/h)$. By definition, $\mathcal{C}(\tilde{\nu})$ corresponds to a Čech cocycle ν/h such that $\nu = \tilde{\nu} \wedge dh$ (take $v = \tilde{\nu},\ w = \nu$ in the above description of ω_D^q with the help of multiplication by the fundamental class). This yields $\mathcal{C}(\tilde{\nu}) = \nu/h = \tilde{\nu} \wedge dh/h$, and $\operatorname{res.}(\nu/h) = \tilde{\nu}$. Thus, for any element $\tilde{\nu} \in \omega_D^q$ there is a preimage under the logarithmic residue map represented by ν/h. This completes the proof.

Remark 6.4 In fact, the representation $(*)$ implies directly that $\operatorname{res.}\Omega_S^m(\log D) \cong \omega_D^n \cong \mathcal{O}_D(dz/dh)$, in view of the formal decomposition $dz/h = (dh/h) \wedge (dz/dh)$. Further, it is not difficult to verify that in the case of plane node of Example 2.1 there is natural isomorphisms $\operatorname{res.}\Omega_S^1(\log D) \cong \pi_*(\mathcal{O}_{\tilde{D}}) \cong \omega_D^0$ (cf. Remark 5.3). A similar result is also valid in a more general situation (see [27], Theorem (2.9)).

Remark 6.5 It should also be underlined that there is a far reaching generalization of main results of this section to the case of complete intersections. In papers [5] and [6] it was developed the theory of *multi-logarithmic* differential forms and their residues with applications to the general theory of multidimensional residue and residue currents on complex spaces.

Bibliography

1. A.G. Aleksandrov. On the De Rham complex of non-isolated singularities. Funktsional. Analiz i ego Prilogeniya 22 (1988), no. 2, 59-60.

2. A.G. Aleksandrov. Nonisolated hypersurface singularities. In: Theory of singularities and its applications. Advances in Soviet Mathematics (Arnold V.I., ed.) 1, 211-246. Amer. Math. Soc., Providence, RI, 1990.

3. A.G. Aleksandrov. Logarithmic differential forms, torsion differentials and residue. Complex Variables, Theory Appl. 50 (2005), No. 7-11, 777-802.

4. A.G. Aleksandrov, S Tanabé. Gauss-Manin connexions, logarithmic forms and hypergeometric functions. In: Geometry from the Pacific Rim (edited by J.Berrick et al.), 1-21. Walter de Gruyter, Berlin – New York, 1997.

5. A.G. Aleksandrov, A.K Tsikh. Théorie des résidus de Leray et formes de Barlet sur une intersection complète singulière, C. R. Acad. Sci. Paris, Sér. I, Mathématique 333 (2001), No. 11, 973-978.

6. A.G. Aleksandrov, A.K. Tsikh. Multi-logarithmic differential forms on complete intersections, Journal of Siberian Federal University, Mathematics & Physics 2 (2008), 105-124.

7. K. Aomoto. Configurations and invariant Gauss-Manin connections of integrals I, Tokyo J.Math. 5 (1982), 249-287

8. D. Barlet. Le faiseau $\omega_X^\bullet$ sur un espace analytique X de dimension pure. Lecture Notes in Mathematics 670, Springer-Verlag, 1978, 187-204.

9. P. Cartier. Les arrangements d'hyperplans: un chapitre de géometrie combinatoire, Lecture Notes in Mathematics 901, 1-22. Springer-Verlag, 1981.

10. F. Castro-Jiménez, D. Mond,L. Narváez-Macarro. Cohomologie du complémentaire d'un diviseur libre. C. R. Acad. Sci. Paris Sér. I Math. 320 (1995), no. 1, 55-58.

11. P. Deligne. Equations différentieles à points singuliers réguliers, Lecture Notes in Mathematics 163, Springer-Verlag, 1970.

12. P. Deligne. Théorie de Hodge. II. Inst. Hautes Études Sci. Publ. Math. 40 (1971), 5-57.

13. I.M. Gelfand, M.M. Kapranov, A.V. Zelevinski Generalized Euler integrals and A-hypergeometric functions, Adv. in Math. 84 (1990), no.2, 255-271.

14. G.M. Greuel. Der Gauß-Manin-Zusammenhang isolierter Singularitäten von vollständigen Durchschnitten, Math. Ann. 214 (1975), no.3, 235-266.

15. Ph. Griffiths. Variations on a theorem of Abel, Invent. Math. 35 (1976), 321-390.

16. Ph. Griffiths, J. Harris. Principles of algebraic geometry. Pure and Applied Mathematics. Wiley-Interscience [John Wiley & Sons], New York, 1978.

17. G.M. Henkin. Jean Leray and several complex variables. In: Selected papers of J.Leray, vol. III, 1-31. Springer-Verlag and SMF, 1998.

18. E. Kunz. Residuen von Differentialformen auf Cohen-Macaulay Varitäten, Math. Zeitschrift 152 (1977), 165-189.

19. Lê Dũng Tráng, K. Saito. The local π_1 of the complement of a hypersurface with normal crossings in codimension 1 is abelian, Ark. Mat. 22 (1984), no. 1, 1–24.

20. Manin, I. Yuri. Frobenius manifolds, quantum cohomology, and moduli spaces,Colloquium Publications. American Mathematical Society (AMS), 47. Providence, RI: American Mathematical Society (AMS), 1999.

21. P. Orlik, H. Terao. Arrangements of hyperplanes. Gründlehren der Mathematischen Wissenschaften 300. Springer-Verlag, Berlin, 1992.

22. F. Pham. Singularitiés des systèmes différentiels de Gauss-Manin, Progress in Mathematics 2, Birkhäuser Verlag, Basel, 1979.

23. C. Peters, J.H.M. Steenbrink. Infinitesimal variations of Hodge structure and the generic Torelli problem for projective hypersurfaces. Progress in Math. 39. Birkhäuser, 1983, 399-464.

24. J.-B. Poly. Sur un théorème de J.Leray en théorie des résidus, C. R. Acad. Sci. Paris Sér. A-B 274 (1972), A171–A174.

25. K. Saito. On the uniformization of complements of discriminant loci. Preprint # S1-KS, Williams College, Williamstown, Mass., 1975, 1-21.

26. K. Saito. On the uniformization of complements of discriminant loci. In: Hyperfunctions and Linear partial differential equations, RIMS Kōkyūroku 287 (1977), 117-137.

27. K. Saito. Theory of logarithmic differential forms. J. Fac. Sci. Univ. Tokyo, ser. IA 27 (1980), no. 2, 265-291.

28. K. Saito, T. Yano, J. Sekiguchi. On a certain generator system of the ring of invariants of a finite reflection group, Comm. Algebra 8 (1980), no. 4, 373-408.

29. J.H.M. Steenbrink. Mixed Hodge structures associated with isolated singularities. In: Singularities, Part 2 (Arcata, Calif., 1981), 513–536, Proc. Sympos. Pure Math., 40, Amer. Math. Soc., Providence, R.I., 1983.

30. O. Zariski. Characterization of plane algebroid curves whose module of differentials has maximum torsion, Proc. Nat. Acad. Sci. U.S.A. 56 (1966), 781-786.

ВЫЧЕТЫ ЛОГАРИФМИЧЕСКИХ ДИФФЕРЕНЦИАЛЬНЫХ ФОРМ

А.Г. Александров

Институт проблем управления РАН,

ул. Профсоюзная, 65, Москва, 117997, Россия, e-mail: ag_aleksandrov@mail.ru

Аннотация. В этой заметке излагается элементарное введение в теорию логарифмических дифференциальных форм и их вычетов. В частности, рассматриваются некоторые свойства логарифмических форм, связанные с кручением голоморфных дифференциалов на особых гиперповерхностях, кратко обсуждаются понятия вычета, данные Пуанкаре, Лерэ и Саито, а затем приводится красивое описание регулярных мероморфных дифференциалов в терминах вычетов мероморфных дифференциальных форм, логарифмических вдоль гиперповерхности с произвольными особенностями.

Ключевые слова: логарифмические дифференциальные формы, форма-вычет, регулярные мероморфные дифференциальные формы, кручение голоморфных дифференциалов.

УДК 515.17

LOGARITHMIC DIFFERENTIAL FORMS AND ANALYSIS OF COMPLEX DYNAMICAL SYSTEMS

A.G. Aleksandrov[1], A.A. Castro, Jr.[2], V.A. Gruzman[1]

[1]Institute for Control Sciences, Russian Academy of Sciences,
Profsojuznaja str., 65, Moscow, 117997, Russian Federation, e-mail: ag_aleksandrov@mail.ru

[2]Departamento de Matemática, Universidade Federal da Bahia Avenida Ademar de Barros,
s/n Campus Universitário de Ondina Salvador, Bahia, 40170-110, Brazil, e-mail: armando@impa.br

Abstract. A wide class of complex dynamical systems can be described by evolutionary processes given by a vector field with polynomial, analytic or smooth coefficients in phase space. Such systems are investigated by perturbation analysis of the control and behavioral spaces together with associated bifurcation sets and discriminants. Our approach is based essentially on the theory of logarithmic differential forms, deformations theory and integrable connections associated with deformations. Such a connection can be represented as a holonomic system of differential equations of Fuchsian type whose coefficients have logarithmic poles along the bifurcation set or discriminant of a deformation. In addition we also describe another interesting application, a new method for computing the topological index of a complex vector field on hypersurfaces with arbitrary singularities.

Keywords: logarithmic differential forms, hypersurface singularity, torsion differentials, regular meromorphic differential forms, residue-map, index of vector fields.

Introduction

Let us consider a complex dynamical system given by an evolutionary process described by a vector field in phase space. A point of phase space defines the state of such system. The vector at this point indicates the velocity of change of the state. The points where the vector field vanishes are called equilibrium points, equilibrium positions or singularities of the vector field.

It was shown by [9] that the typical phase portraits in the neighbourhood of an equilibrium point of a generic system can be classified so that the corresponding list consists of the five simple types: two stable (focus, node) and three unstable (saddle, focus, node).

Of course, generic systems or, in other words, systems which are in general position correspond to real evolutionary processes and vice versa. Such a system always depends on parameters that are never known exactly. A small generic change of parameters transforms a non-generic system into a generic one. Thus, at the first sight, more complicated cases might not be considered since they turn into combinations of the above types after a small generic perturbation of the system.

However, if one is interested not in an individual system but in systems depending on parameters the situation is quite different and more complex. Thus, let us consider the space

A.G. Aleksandrov partially supported by the Russian Foundation of Basic Research (RFBR) and by the National Natural Science Foundation of China (NSFC) in the framework of the bilateral project "Complex Analysis and its applications"(project No. 08-01-92208), and by the National Counsel of Technological and Scientific Development CNPq of Brazil (project No. 454594/2007-0)

A.A. Castro, Jr. partially supported by the National Counsel of Technological and Scientific Development CNPq of Brazil (project No. 454594/2007-0)

of all systems divided into domains of generic systems. The dividing sets (hypersurfaces) correspond to degenerate systems. Under a small change of the parameters a degenerate system becomes non-degenerate. A one-parameter family of systems is presented by a curve which can intersect transversely the boundary separating different domains of nondegenerate systems.

Hence, although for each fixed value of the parameter the system can be always transformed by a small perturbation into a nondegenerate one, it is impossible to do this simultaneously for *all* values of the parameter. In fact, every curve closed to the one considered intersects the boundary of the separate hypersurface at a close enough value of the parameter.

Thus, if one studies not an individual system only but the whole family, the degenerate cases are not *removable*. If the family depends on a one parameter than the simplest degeneracies are unremovable one, those represented by boundaries of codimension one (that is, boundaries given by one equation) in the space of all systems. The more complicated degenerate systems, forming a set of codimension two in the space of all systems, may be gotten rid of by a small perturbation of the one-parameter family.

If one analyzes two-parameter families then one needs not to consider degenerate systems forming a set of codimension three and so on. Therefore at first it ought to analyze all generic systems, then degeneracies of codimension one, then – two and so on (see [4]). Herewith one must not restrict the study of degenerate systems to the picture at the moment of degeneracy, but must also include a description of the reorganizations that take place when the parameter passes through the degenerate value.

1 Control space and parameters

Let us consider a family of *smooth* functions

$$f : \mathbf{R}^n \times \mathbf{R}^r \to \mathbf{R},$$

describing a certain process happening in various copies of $\mathbf{R}^n$ governed by the function f and affected by the point in $\mathbf{R}^r$. The coordinate space $\mathbf{R}^n$ is usually called the space of *internal* variables while $\mathbf{R}^r$ the space of *external* variables over which each copy sits. Such terminology is suitable when the variables in $\mathbf{R}^r$ label in physical space as in mechanics, optics, biology or ecology, and so on.

For systems which one alters something and then to observe what happens the variables in $\mathbf{R}^r$ are called the *control* parameters while the variables in $\mathbf{R}^n$ are called the *behavioral* parameters. Accordingly the space $\mathbf{R}^r$ is referred to as the *control* space while $\mathbf{R}^n$ as the *behavior* space. In the strictly mathematical context it natural to call the space $\mathbf{R}^r$ the *deformation space* while its points (or their coordinates) the *parameters of a deformation*. The number r is correspondingly the *external* or *control* dimension, or the *dimension* of deformation.

Suppose that a submanifold $M \subset \mathbf{R}^n \times \mathbf{R}^r$ is given by the equation

$$\mathcal{D} f_u(x) = 0,$$

where $f_u(x) = f(x,u),\ (x,u) \in \mathbf{R}^n \times \mathbf{R}^r$, and $\mathcal{D}$ is the usual differential of the image

$$f_u : \mathbf{R}^n \longrightarrow \mathbf{R}.$$

In other words, the manifold M is the set of all *critical points* of all the potentials f_u in the family f. Denote by ξ the restriction to M of the natural projection

$$\pi : \mathbf{R}^n \times \mathbf{R}^r \longrightarrow \mathbf{R}^r, \qquad \pi(x,u) = u.$$

The *critical set* is identified with the subset $\mathcal{C} \subset M$ consisting of *singular* points of the image ξ. In other words, $\mathcal{C}$ consists of points in which the map ξ is *singular*, that is, the rank of the derivative $\mathcal{D}\xi$ is less than r. The image of the critical set $\xi(\mathcal{C}) \subset \mathbf{R}^r$ is called the *bifurcation set* $\mathcal{B}$.

It is not difficult to see, by computing $\mathcal{D}\xi$, that $\mathcal{C}$ is the set of points $(x, u) \in M$, at which $f_u(x)$ has a *degenerate critical* point. It follows that $\mathcal{B}$ is the locus where the number and nature of critical points *change* (that is, it happens jump changes in the state of a control system); for by structural stability of Morse functions such changes can only occur by *passing* through a degenerate critical point. In most applications (for instance, in problems of stability, optimization, in studying caustics, wave fronts and so on) it is the *bifurcation* set that is the most important, for it lies in the control space, hence is "observable and all delay convention jumps occur in it.

Investigations show that a bifurcation set as variety possesses highly complicated topological, analytic, algebraic and geometric *structures*. Herewith it appears that characteristics of such a variety depend mainly on the structure of its subvariety of *singularities*, which, in turn, also can possess singularities and so on. This observation directly leads to the notion of a *stratification* variety, but in the most general context the study of bifurcation sets is reduced to the study of *stratified* varieties (see [11]).

Remark that in virtue of the well-known splitting lemma a *smooth* function f can be represented around a point, where it has corank k, in the form:

$$\widetilde{f}(x_1, \ldots, x_k) \pm x_{k+1}^2 \pm \ldots \pm x_n^2$$

(perhaps with parameters in $\mathbf{R}^r$ for $\widetilde{f}$). Herewith the variables $x_1, \ldots, x_k$ are called *essential* while $x_{k+1}, \ldots, x_n$ – *unessential*. Certainly, such presentation is very far from unique. It should be also noted that most singularities met by an r-dimensional family will even when not regular or Morse, have codimension less than r. However, it is possible to write an r-parameter family f, around a point, where it meets transversely a singularity of codimension ν, in a way in which only ν control parameters appear. When one has done so, one may call the coordinates on $\mathbf{R}^r$ that no longer appear, *disconnected* or *dummy* control parameters.

2 Deformations

In fact, general evolutionary processes can be described with the help of polynomial, analytical or smooth functions and systems of equations as well as in a wider context by systems of differential equations. In particular, using properties of associated bifurcation sets, the discriminants or, more generally, singular loci, basic properties of the corresponding systems are investigated. One of the most efficient tools of the investigation is a general notion of integrable connection associated naturally with any deformation of a system. Let us shortly discuss basic ideas of the theory. Consider the system of polynomial or analytic equations

$$\left\{ \begin{array}{rcl} f_1(z_1, z_2, \ldots, z_m) & = & 0 \\ & \vdots & \\ f_k(z_1, z_2, \ldots, z_m) & = & 0 \end{array} \right. \tag{2.1}$$

given in a neighbourhood U of the origin in $\mathbf{C}^m$. For simplicity we shall assume that $k = m - 1$ and the set X_0 of the solutions of our system in the neighbourhood U is one-dimensional. We shall say that a point, laying on the curve X_0, is *nonsingular* if the differential form $df_1 \wedge \ldots \wedge df_{m-1}$ does not vanish on it. Otherwise, this point (and the curve) refers to as *singular*, or shortly a *singularity*. Without loss of generality one can suppose that X_0 has the only singularity at the origin $\{0\} \in X_0 \subset \mathbf{C}^m$, that is, X_0 is the germ of a reduced *curve.*

We shall assume now that the equations of the system (2.1) can be *perturbed:*

$$\begin{cases} f_1(z_1, z_2, \ldots, z_m) & = & t_1 \\ \vdots & & \\ f_{m-1}(z_1, z_2, \ldots, z_m) & = & t_{m-1} \end{cases} \tag{2.2}$$

in such a manner that at each sufficiently small value of parameters $t = (t_1, \ldots, t_{m-1}) \in \mathbf{C}^{m-1}$ in the chosen neighbourhood U the set X_t of the solutions of the system (2.2) is also *one-dimensional.* In other words, we shall consider the *principal (flat) deformation* of the curve singularity X_0 given by the holomorphic map:

$$f\colon \mathcal{X} \longrightarrow \mathbf{C}^{m-1}. \tag{2.3}$$

Let X be the intersection of a ball of a small radius $\varepsilon > 0$ centered at the distinguished point $\{0\} \in X_0$ with $f^{-1}(T)$, where $T \subset \mathbf{C}^{m-1}$ is a punctured ball of a radius $0 < \delta \ll \varepsilon$ centered at the origin $0 = f(0)$. Consider the natural restriction $f\colon X \longrightarrow T$ of the mapping (2.3). Then for some values of parameters $t \in T$ the fibres X_t are non-singular curve germs, for other ones the corresponding fibres may have singular points called the *critical points* of map f.

3 Period integrals

Denote by $C \subset X$ the set of critical points of f and by D its image $f(C) \subset T$. Thus, parameters corresponding to the fibres with singularities form the set D which refers to as the discriminant set, or the *discriminant* of the principal deformation X_0. In many important cases the discriminant is the zero-set of the only equation $h(t_1, \ldots, t_k) = 0$, that is, D is a *hypersurface.* Set

$$T' = T - D, \quad X' = X - C.$$

The restriction $f\colon X' \to T'$ is a local trivial differentiable fibre bundle called the *Milnor fibration* of f, that is, fibres $X_t = f^{-1}(t) \cap X$ (of *real* dimension two) form a smooth fibre bundle over T'. Fix a point $t_0 \in T'$. Then for each *smooth closed path* $\gamma_0 \subset X_{t_0}$, corresponding to the 1-cycle in $H^1(X_{t_0}, \mathbf{C})$, it is possible to construct a family of 1-cycles $\gamma(t) \subset X_t,\ t \in T'$, such that $\gamma(t_0) = \gamma_0$.

If one takes a holomorphic differential form $\omega = g(z) dz_1 \wedge \ldots \wedge dz_m$ of the maximal degree in a neighbourhood of the origin in $\mathbf{C}^m$, then using the identity $df_1 \wedge \ldots \wedge df_{m-1} \wedge \psi = \omega$ one can find a differential form ψ, which is the result of the division of ω by $df_1 \wedge \ldots \wedge df_{m-1}$. The form ψ is not determined uniquely, but up to the summands containing differentials of the functions $f_1, \ldots, f_{m-1}$. It is easy to prove, that for all parameters t, rather close to zero, the integral

$$I(t) = \int_{\gamma(t)} \psi = \int_{\gamma(t)} \frac{\omega}{df_1 \wedge \ldots \wedge df_{m-1}} \tag{3.1}$$

is determined correctly. Moreover, the integral $I(t)$ is an *analytic* function in the variable t. Integrals of such type are called the *period integrals.*

Replacing the differential form ω with another, the integral (3.1), generally speaking, will also change. However, it is possible to prove that the set of all such integrals contains a finite number of the elements $I_1(t), \dots, I_\mu(t)$ so that any integral of the type (3.1) may be expressed by means of these generators as a linear combination with *holomorphic* coefficients. In the present context μ is the *Milnor number* which is a topological invariant of the singularity X_0.

The same observation holds, if one fixes the form ω and takes various families $\gamma(t)$. For definiteness, we shall fix a family of vanishing cycles $\gamma(t)$ and consider μ independent *period integrals* of the following type

$$I_j(t) = \int_{\gamma(t)} \frac{\omega_j}{df_1 \wedge \ldots \wedge df_{m-1}},$$

where $1 \le j \le \mu$. The period integrals $I_j(t)$ can be differentiated with respect to the parameter t. Between integrals and their derivatives there arose linear relations *(syzygies)* with polynomial coefficients in t. These relations generate a system of differential equations for the integrals $I_j(t)$ expressed through a finite number of independent integrals.

4 Connection

In such a way a system of differential equations in the variable t is associated with the germ X_0; this system is defined correctly outside of the discriminant and refers to as *Gauss-Manin connection,* or *Gauss-Manin system,* associated with the principal deformation of X_0. The main problem is to describe a system of differential equations defined on the *whole* space of parameters, which is equivalent to the initial one outside of the discriminant (or, in other words, to *extend* the initial system to the discriminant set). It is possible to show that the solution of this problem depends mainly on properties of the discriminants as well as on properties of fibres of the deformation.

It turns out that the connection in question can be represented in a quite elegant form. In order to explain this idea we need the following notion. Let ω be a meromorphic differential form on S having poles along a reduced divisor $D \subset S$. Then ω is called the *logarithmic* along D differential form if and only if ω and its total differential $d\omega$ have poles along D at worst of the first order. That is, $h\omega$ as well as $hd\omega$ are *holomorphic* differential forms on S where h is a local equation of the hypersurface $D \subset S$.

The $\mathcal{O}_S$-module of logarithmic differential q-forms is usually denoted by $\Omega^q_S(\log D)$. Logarithmic differential forms have many remarkable analytic and algebraic properties (for example, see [1]).

Following [10] denote by $\mathrm{Der}_S(\log D)$ the $\mathcal{O}_S$-module of *logarithmic vector fields* along D on S. This module consists of germs of holomorphic vector fields η on S for which $\eta(h)$ belongs to the principal ideal $(h)\mathcal{O}_S$. In particular, the vector field η is tangent to D at its smooth points. The inner multiplication of vector fields and differential forms induces a natural pairing of $\mathcal{O}_S$-modules

$$\mathrm{Der}_S(\log D) \times \Omega^q_S(\log D) \longrightarrow \Omega^{q-1}_S(\log D).$$

For $q = 1$ this $\mathcal{O}_S$-bilinear mapping is a *non-degenerate* pairing so that $\mathrm{Der}_S(\log D)$ and $\Omega^1_S(\log D)$ are $\mathcal{O}_S$-dual.

Let $\mathcal{H}$ be a free $\mathcal{O}_S$-module. Then a connection ∇ on $\mathcal{H}$ with *logarithmic* poles along $D \subset S$ is a $\mathbf{C}$-linear morphism

$$\nabla_{X/S}\colon\ \mathcal{H} \longrightarrow \mathcal{H} \otimes_{\mathcal{O}_S} \Omega^1_S(\log D) \tag{4.1}$$

satisfying the following conditions:

$$\begin{aligned} &1)\ \nabla(\omega+\omega')=\nabla(\omega)+\nabla(\omega'),\\ &2)\ \nabla(f\omega)=\omega\otimes df+f\nabla(\omega),\quad f\in\mathcal{O}_S. \end{aligned}$$

Consider the case where $\Omega^1_S(\log D)$ is a *free* $\mathcal{O}_S$-module of rank m. Obviously, in such a case $\Omega^p_S(\log D)=\bigwedge^p\Omega^1_S(\log D)$, $p\geq 1$. It is often said that the divisor D is *free* or, equivalently, D is a *Saito free* divisor. The following characteristic property of such divisors was discovered by [10].

Proposition 4.1 *Suppose that there exist m logarithmic vector fields $\mathcal{V}^1,\dots,\mathcal{V}^m\in\mathrm{Der}_S(\log D)$ such that for the $(m\times m)$-matrix $\mathcal{M}$ whose entries are the coefficients of $\mathcal{V}^i$, $i=1,\dots,m$, one has $\det(\mathcal{M})=ch$, where c is a unit. Then $\mathcal{V}^1,\dots,\mathcal{V}^m$ form a basis of the free $\mathcal{O}_S$-module $\mathrm{Der}_S(\log D)$. In particular, $\Omega^1_S(\log D)$ is a free $\mathcal{O}_S$-module with the dual basis $\omega_1,\dots,\omega_m$.*

For example, $\Omega^1_S(\log D)$ is free when D is the discriminant of the *minimal versal* deformation of the system defined by a function with isolated singularity.

Now let D be a Saito free divisor. Then we can describe the *logarithmic connection* (4.1) on $\Omega^1_S(\log D)$ itself. In other words, let us consider the case when $\mathcal{H}=\Omega^1_S(\log D)$:

$$\nabla\colon\ \Omega^1_S(\log D)\to\Omega^1_S(\log D)\otimes_{\mathcal{O}_S}\Omega^1_S(\log D).$$

Let $\omega_1,\dots,\omega_m$ be free generators of the module $\Omega^1_S(\log D)$. Then the connection ∇ can be expressed in terms of Christoffel symbols in the following way:

$$\nabla\omega_i=\sum_{j=1}^m\omega_j\otimes\omega_i^j,\quad \omega_i^j=\sum_{k=1}^m\Gamma_i^{jk}\omega_k.$$

The connection ∇ is called *torsion free* if

$$d\,\omega_i=\sum_{j=1}^m\omega_i^j\wedge\omega_j=\sum_{k,j=1}^m\Gamma_i^{jk}\omega_k\wedge\omega_j,$$

and ∇ is called *integrable* if

$$d\omega_i^j=\textstyle\sum_{k=1}^m\omega_i^k\wedge\omega_k^j,\quad \text{that is,}\quad d\nabla=\nabla\wedge\nabla,$$

where $\nabla=\|\omega_i^j\|$ is the *coefficient matrix* of the connection ∇. In particular, it means that the composition

$$\mathcal{H}\xrightarrow{\nabla}\mathcal{H}\otimes\Omega^1_S(\log D)\xrightarrow{\nabla}\mathcal{H}\otimes\wedge^2\Omega^1_S(\log D)$$

is zero.

5 Holonomic systems

It is possible to associate with any integrable and torsion free connection ∇ on the module $\Omega^1_S(\log D)$ a holonomic system of Fuchsian type in the following way.

It is known (see [1]) that the multiplication by h induces the surjection

$$\Omega^1_S(\log D) \xrightarrow{\cdot h} \operatorname{Tors}\Omega^1_D \longrightarrow 0, \tag{5.1}$$

whose kernel coincides with an $\mathcal{O}_S$-module

$$\mathcal{O}_S\frac{dh}{h} + \Omega^1_S.$$

Here Ω^1_S is the module of holomorphic differential 1-forms on S generated by the differentials $dz_1, \dots, dz_m$ over $\mathcal{O}_S$,

$$\Omega^1_D = \Omega^1_S/(h\,\Omega^1_S + \mathcal{O}_S dh)$$

is the module of regular differential 1-forms on the divisor D, and $\operatorname{Tors}\Omega^1_D$ is the *torsion* submodule of Ω^1_D. The support of $\operatorname{Tors}\Omega^1_D$ is contained in the singular locus $\operatorname{Sing} D$ of the hypersurface D. The torsion $\mathcal{O}_D$-module $\operatorname{Tors}\Omega^1_D$ has a system of generators containing at least $m-1$ elements.

By definition, the generalized Fuchsian system is a holonomic system of linear differential equations on S with *meromorphic* coefficients containing in $\Omega^1_S(\log D)$:

$$d\,\mathbf{I} = \Omega\ \mathbf{I}, \tag{5.2}$$

where $\mathbf{I} = {}^{\mathrm{tr}}(I_1, \dots, I_k)$ is a vector-column of unknown functions and the matrix differential form Ω is defined as follows:

$$\Omega = A_0\frac{dh}{h} + \sum_{i=1}^{\ell} A_i\frac{\vartheta_i}{h}.$$

Here the differential 1-forms $\vartheta_i \in \Omega^1_S,\ i = 1, \dots, \ell$, correspond via (5.1) to non-zero elements of the torsion submodule $\operatorname{Tors}\Omega^1_D$, and $A_i \in \operatorname{End}(\mathbf{C}^k)\otimes\mathcal{O}_S,\ i = 0, 1, \dots, \ell$, are coefficient matrices with holomorphic entries such that the integrability condition $d\,\Omega = \Omega\wedge\Omega$ holds.

It is not difficult to show that one can associate to any integrable and torsion free connection ∇ on the module $\Omega^1_S(\log D)$ the generalized Fuchsian system of type (5.2) (see [3]). Moreover, using the Christoffel symbols of such connection, it is possible to express the integrability condition in terms of commuting relations of the coefficient matrices $A_i,\ i = 1, \dots, \ell$.

Under some additional assumptions on entries of the coefficient matrices A_i it is possible to investigate the system of type (5.2) and to describe its explicit solutions. In fact, such solutions are quite useful in describing the control of evolutionary processes, perturbations of multidimensional systems, and many applications in dynamical systems, bifurcation theory, etc. (for example, see [8], [4]).

6 Topological index

The index of a vector field is one of the very first concepts in topology and geometry of smooth manifolds, and its properties underlie important results of the theory, including the Poincaré-Hopf theorem, which states that the total index of a vector field on a closed smooth orientable manifold is independent of the field and coincides with the Euler-Poincaré characteristic of the manifold. When studying singular varieties such as bifurcation sets, discriminants, etc., it is natural to ask whether there exists a similar invariant in a more general context. One possible generalization of this type, which originally arose in topology of foliations, turned out to be well suited for use in the theory of singular varieties. In this section, we shortly describe a new method for the calculation of the index of vector fields on a hypersurface on the basis of the theory of logarithmic differential forms and vector fields. The main idea of our approach is to describe the index in terms of meromorphic differential forms defined on the ambient variety and having logarithmic poles along the hypersurface (see [2]). We shall see that the systematic use of the theory of logarithmic forms permits one not only to simplify the calculations dramatically but also to clarify the meaning of the basic constructions underlying many papers on the subject (for example, see [6]).

6.1 Regular differential forms

Let S be a complex manifold of dimension $m = n+1$, $n \geq 1$, and let Ω^q_D be the $\mathcal{O}_D$-module of germs of regular (Kähler) differentials of order q on D, so that

$$\Omega^q_{D,x} = \Omega^q_{S,x} / \big(h \cdot \Omega^q_{S,x} + dh \wedge \Omega^{q-1}_{S,x}\big), \quad q \geq 0,$$

where $x \in S$. By analogy with smooth case, elements of $\Omega^q_{D,x}$ are usually called germs of regular holomorphic forms on D. Now let $\mathrm{Der}(D) = \mathrm{Hom}_{\mathcal{O}_D}(\Omega^1_D, \mathcal{O}_D)$ be the sheaf of germs of regular vector fields on D and let us consider an element $V \in \mathrm{Der}(D)$. By $\mathcal{V} \in \mathrm{Der}(S)$ we denote a holomorphic vector field on S such that $\mathcal{V}|_D = V$. Then the interior multiplication (contraction) $\iota_{\mathcal{V}} \colon \Omega^q_S \longrightarrow \Omega^{q-1}_S$ of vector fields and differential forms defines the structure of a complex on $\Omega^\bullet_S$, since $\iota^2_{\mathcal{V}} = 0$. The contraction $\iota_{\mathcal{V}}$ induces a homomorphism $\iota_V \colon \Omega^q_D \longrightarrow \Omega^{q-1}_D$ of $\mathcal{O}_D$-modules and also the structure of a complex on $\Omega^\bullet_D$. The corresponding ι_V-homology sheaves and groups are denoted by $H_*(\Omega^\bullet_D, \iota_V)$ and $H_*(\Omega^\bullet_{D,x}, \iota_V)$, respectively.

6.2 Homological index

If the vector field $\mathcal{V}$ has an *isolated* singularity at a point $x \in D$, then ι_V-homology groups of the complex $\Omega^\bullet_{D,x}$ are finite dimensional vector spaces, so that the Euler characteristic

$$\chi\big(\Omega^\bullet_{D,x}, \iota_V\big) = \sum_{i=0}^{n+1} (-1)^i \dim H_i\big(\Omega^\bullet_{D,x}, \iota_V\big),$$

of the complex of regular differentials is well-defined. It is called the *homological* index of the vector field V at the point $x \in D$ and denoted by $\mathrm{Ind}_{\mathrm{hom},D,x}(V)$ (see [7]). At *nonsingular* points of D the homological index coincides with the *topological* index, or, equivalently, with the Poincaré-Hopf local index.

6.3 Logarithmic index

Let us consider a vector field $\mathcal{V} \in \mathrm{Der}_S(\log D)$. The interior multiplication $\iota_{\mathcal{V}}$ defines the structure of a complex on $\Omega^\bullet_S(\log D)$.

Lemma 6.1 *If all singularities of the vector field $\mathcal{V}$ are isolated, then $\iota_{\mathcal{V}}$-homology groups of the complex $\Omega^\bullet_S(\log D)$ are finite dimensional vector spaces.*

Proof. Assume that $S \cong \mathbf{C}^m$, $m = n+1$, and the point $x_0 = 0 \in D \subset S$ is an isolated singularity of the field $\mathcal{V}$, so that $\mathcal{V}(x_0) = 0$. Then $\mathcal{V}(x) \neq 0$ at each point x in a sufficiently small punctured neighbourhood of x_0. In a suitable neighbourhood of x there exists a coordinate system $(t, z'_1, \dots, z'_n)$ such that $\mathcal{V} = \partial/\partial t$. Since $\mathcal{V}(h) \subseteq (h)\mathcal{O}_{S,0}$, it follows that $D \cong T \times D_0$, where T is a small disc in the variable t and D_0 is a hypersurface in $\mathbf{C}^n$. It is easy to show that

$$\Omega^q_{\mathbf{C}^{n+1},0}(\log D) \cong \big(\Omega^q_{\mathbf{C}^n,0}(\log D_0) \oplus \Omega^{q-1}_{\mathbf{C}^n,0}(\log D_0) \wedge dt\big) \otimes_{\mathbf{C}} \mathcal{O}_{\mathbf{C},0}.$$

Indeed, for germs of holomorphic forms one has the isomorphism $\Omega^q_{D,0} \cong \big(\Omega^q_{D_0,0} \oplus \Omega^{q-1}_{D_0,0} \wedge dt\big) \otimes_{\mathbf{C}} \mathcal{O}_{\mathbf{C},0}$ which can readily be obtained by considering of the canonical projections of the analytic set $T \times D_0$ onto the first and second factors and the definition of $\Omega^q_{D,0}$. The desired isomorphism for germs of logarithmic forms can be obtained by a similar argument with the use of the exact sequence

$0 \to \Omega^q_{\mathbf{C}^{n+1},0} + \frac{dh}{h} \wedge \Omega^{q-1}_{\mathbf{C}^{n+1},0} \to \Omega^q_{\mathbf{C}^{n+1},0}(\log D) \xrightarrow{\cdot h} \Omega^q_{D,0} \to \Omega^q_{D,0}/\operatorname{Tors}\Omega^q_{D,0} \to 0,$ which follows from the exact sequence expressing the torsion subsheaves $\operatorname{Tors}\Omega^q_D$ in terms of logarithmic differential forms (see [2]).

Further, in the q-th piece of the complex $(\Omega^\bullet_{S,0}(\log D), \iota_{\mathcal{V}})$ one has

$$\operatorname{Ker}(\iota_{\partial/\partial t}) \cong \operatorname{Im}(\iota_{\partial/\partial t}) \cong \big(\Omega^q_{\mathbf{C}^n,0}(\log D_0) \oplus (0)\big) \otimes_{\mathbf{C}} \mathcal{O}_{\mathbf{C},0}.$$

That is, the corresponding homology groups vanish for all q. The same conclusion readily follows for the point $x_0 \in S \setminus D$. Consequently the $\iota_{\mathcal{V}}$-homology groups of the complex $\Omega^\bullet_S(\log D)$ may be non-trivial only at singular points of the field. Since the sheaves of logarithmic forms as well as their cohomology are coherent, we arrive at the statement of the Lemma.

Thus if the vector field $\mathcal{V}$ has *isolated singularities,* then the Euler characteristic

$$\chi\big(\Omega^\bullet_{S,x}(\log D), \iota_{\mathcal{V}}\big) = \sum_{i=0}^{n+1} (-1)^i \dim H_i\big(\Omega^\bullet_{S,x}(\log D), \iota_{\mathcal{V}}\big)$$

of the complex of logarithmic differential forms is well defined for any point $x \in S$. It is called the *logarithmic* index of the field $\mathcal{V}$ at the point x and denoted by $\operatorname{Ind}_{\log D,x}(\mathcal{V})$. It follows from the preceding that $\operatorname{Ind}_{\log D,x}(\mathcal{V}) = 0$ whenever $\mathcal{V}(x) \neq 0$.

6.4 The index of vector fields on hypersurfaces

To study the $\iota_{\mathcal{V}}$-homology of the complex $\Omega^\bullet_D$, one can use an approach based on a representation of regular holomorphic differential forms on the hypersurface D via meromorphic forms with logarithmic poles along D (see [2]). Recall [loc. cite] that for all $q = 0, 1, \dots, n+1$, there exist exact sequences

$$0 \to \Omega^{q-1}_{S,x}/h \cdot \Omega^{q-1}_{S,x}(\log D) \xrightarrow{\wedge dh} \Omega^q_{S,x}/h \cdot \Omega^q_{S,x} \to \Omega^q_{D,x} \to 0$$

of $\mathcal{O}_{S,x}$-modules, where $\wedge dh$ is the homomorphism of exterior multiplication. Hence one obtains the exact sequence

$$0 \to \big(\Omega^\bullet_{S,x}/h\Omega^\bullet_{S,x}(\log D), \iota_{\mathcal{V}}\big)[-1] \xrightarrow{\wedge dh} \big(\Omega^\bullet_{S,x}/h\Omega^\bullet_{S,x}, \iota_{\mathcal{V}}\big) \to \big(\Omega^\bullet_{D,x}, \iota_V\big) \to 0 \tag{6.1}$$

of complexes. Indeed, the fact that the multiplication by $\wedge dh$ induces a morphism of complexes follows from the identity

$$\iota_{\mathcal{V}}(\omega) \wedge dh = \iota_{\mathcal{V}}(\omega \wedge dh) + (-1)^{q-1}\omega \wedge \mathcal{V}(h),$$

since the second summand from the right-hand side vanishes in the quotient complex $\Omega^{\bullet}_S/h\Omega^{\bullet}_S$ in view of the condition $\mathcal{V}(h) \in (h)\mathcal{O}_{S,x}$. Now note that from the exact sequence

$$0 \longrightarrow (\Omega^{\bullet}_{S,x}, \iota_{\mathcal{V}}) \xrightarrow{\cdot h} (\Omega^{\bullet}_{S,x}, \iota_{\mathcal{V}}) \longrightarrow (\Omega^{\bullet}_{S,x}/h\Omega^{\bullet}_{S,x}, \iota_{\mathcal{V}}) \longrightarrow 0$$

of complexes it follows that $\chi\big(\Omega^{\bullet}_{S,x}/h\Omega^{\bullet}_{S,x}, \iota_{\mathcal{V}}\big) = 0$. Thus from the exact sequence (6.1) one obtains

$$\mathrm{Ind}_{\mathrm{hom},D,x}(V) = -\chi((\Omega^{\bullet}_{S,x}/h\Omega^{\bullet}_{S,x}(\log D), \iota_{\mathcal{V}})[-1]) = \chi(\Omega^{\bullet}_{S,x}/h\Omega^{\bullet}_{S,x}(\log D), \iota_{\mathcal{V}}).$$

Proposition 6.2 *Suppose that $x \in D$ is an isolated singularity of a vector field $\mathcal{V} \in \mathrm{Der}(\log D)$, the germs $v_i \in \mathcal{O}_{S,x}$ are determined by the expansion $\mathcal{V} = \sum_i v_i\partial/\partial z_i$, and $J_x\mathcal{V} = (v_1, \dots, v_m)\mathcal{O}_{S,x}$. Then*

$$\mathrm{Ind}_{\mathrm{hom},D,x}(V) = \dim \mathcal{O}_{S,x}/J_x\mathcal{V} - \mathrm{Ind}_{\log D,x}(V).$$

Let us consider the case when D is a Saito free divisor. Then the complex $\big(\Omega^{\bullet}_{S,x}(\log D), \iota_{\mathcal{V}}\big)$ is naturally isomorphic to the *Koszul* complex $\mathbf{K}_{\bullet}\big((\alpha_1, \dots, \alpha_m); \mathcal{O}_{S,x}\big)$ on the generators $e_i = \omega_i,\ i = 1, \dots, m$, where the germs $\alpha_i \in \mathcal{O}_{S,x}$ are determined as coefficients of the expansion $\mathcal{V} = \sum_i \alpha_i\mathcal{V}^i$ of $\mathcal{V}$ in the basis of logarithmic vector fields. In this case one readily obtains the following identity:

$$\mathrm{Ind}_{\log D,x}(\mathcal{V}) = \chi\big(\mathbf{K}_{\bullet}\big((\alpha_1, \dots, \alpha_m); \mathcal{O}_{S,x}\big)\big).$$

Corollary 6.3 *Let $J_{\log D,x}\mathcal{V} = (\alpha_1, \dots, \alpha_m)\mathcal{O}_{S,x}$. Suppose that the coefficients $(\alpha_1, \dots, \alpha_m)$ form a regular $\mathcal{O}_{S,x}$-sequence. Then*

$$\mathrm{Ind}_{\mathrm{hom},D,x}(\mathcal{V}) = \dim \mathcal{O}_{S,x}/J_x\mathcal{V} - \dim \mathcal{O}_{S,x}/J_{\log D,x}\mathcal{V}.$$

6.5 Normal hypersurfaces

Let $Z = \mathrm{Sing}\, D$ be the singular locus of a reduced divisor D, and let $c = \mathrm{codim}\,(Z, D)$ be the codimension of Z in D. It is well-known (see [1]) that $c = 1$ for Saito free divisors, that is, in a sense, the singularities of D form the maximal possible subset of the divisor. For $c \geq 2$, Serre's criterion implies that the hypersurface D is a *normal* variety. For further analysis of this case we use the following reformulation due to [10] of the notion of logarithmic forms.

Lemma 6.4 *The germ ω of a meromorphic differential q-form at a point $x \in S$ with poles along D is the germ of a logarithmic form (that is, $\omega \in \Omega^q_{S,x}(\log D)$) if and only if when there exists a holomorphic function germ $g \in \mathcal{O}_{S,x}$, a holomorphic $(q-1)$-form germ $\xi \in \Omega^{q-1}_{S,x}$ and a holomorphic q-form germ $\eta \in \Omega^q_{S,x}$, such that* (i) $\dim_{\mathbf{C}} D \cap \{z \in M \colon g(z) = 0\} \leq n-1$, (ii) $g\omega = \frac{dh}{h} \wedge \xi + \eta$.

Let $\varrho\colon \Omega^q_{S,x}(\log D) \longrightarrow \Omega^q_{S,x}$ be the homomorphism of multiplication by h, and let $\omega \in \Omega^q_{S,x}(\log D)$. Then there exists an element $g \in \mathcal{O}_{S,x}$ in Lemma 6.4 such that $gh\omega \in h\Omega^q_{S,x} + dh\wedge\Omega^{q-1}_{S,x}$, that is, $gh\omega = 0$ in $\Omega^q_{D,x}$. Since the germ g defines a zero non-divisor in $\mathcal{O}_{D,x}$, in particular, this means that $h\omega \in \operatorname{Tors}\Omega^q_{D,x}$, where the torsion submodule of the sheaf of regular q-differentials is denoted by $\operatorname{Tors}\Omega^q_{D,x}$. Thus, $\operatorname{Im}(\varrho) \subseteq \operatorname{Tors}\Omega^q_{D,x}$ (actually one has the equality). If $\operatorname{Tors}\Omega^q_{D,x} = 0$, then the germ g in (ii) can only be an invertible element; consequently,

$$\Omega^q_{S,x}(\log D) \cong \Omega^q_{S,x} + \frac{dh}{h} \wedge \Omega^{q-1}_{S,x}. \tag{6.2}$$

In fact, this isomorphism can be obtained without the preceding argument if one directly makes use of the exact sequence for the torsion submodules $\operatorname{Tors}\Omega^q_{D,x}$ (for example, see [1]).

Theorem 6.5 *Let D be a normal hypersurface. Then*

$$\operatorname{Ind}_{\mathrm{hom},D,x}(V) = \dim \mathcal{O}_{S,x}/(h, J_x\mathcal{V}) + \textstyle\sum_{i=c'}^{n+1}(-1)^i \dim H_i(\Omega^\bullet_{D,x}, \iota_V),$$

where $c' = 2[\frac{c+1}{2}] + 1$, the square brackets denote the integer part of rational numbers, and the sum is zero by convention if the lower limit is greater than the upper limit.

Proof. It is well-known that $\operatorname{Tors}\Omega^q_D = 0$ if $0 < q < c$. Hence, together with the isomorphism (6.2), this means that $\Omega^q_{S,x}/h\Omega^q_{S,x}(\log D) \cong \Omega^q_{D,x}$ for all such q. Therefore, it follows from the exact sequence (6.1) that

$$H_i(\Omega^q_D, \iota_V) \cong H_{i-1}(\Omega^q_D[-1], \iota_V) = H_{i-2}(\Omega^q_D, \iota_V)$$

for all $i = 3, \dots, c+1$. In particular, in this range the dimensions of the ι_V-homology groups of the complex $\Omega^\bullet_{D,x}$ in the two series H_{2i} and H_{2i-1} coincide. Further, one can readily see that the dimensions of groups H_1 and H_2 also coincide (see [2]), whence the desirable formula follows. The integer part in the lower limit of the sum is needed in order to distinguish between the cases of even and odd codimension.

Corollary 6.6 *Suppose that a point $x \in D$ is an isolated singularity of the hypersurface D as well as of a vector field $\mathcal{V} \in \operatorname{Der}(\log D)$, $\mathcal{V}(h) = \varphi h$ and $\varphi \in \mathcal{O}_{S,x}$. Then*

$$\operatorname{Ind}_{\mathrm{hom},D,x}(V) = \dim \mathcal{O}_{S,x}/(h, J_x\mathcal{V}) + \varepsilon \dim \operatorname{Ann}_{\mathcal{B}_x}(h)/(\varphi)\mathcal{B}_x,$$

where $\varepsilon = -1$ if n is even and $\varepsilon = 0$ otherwise, and $\mathcal{B}_x$ is the local ring $\mathcal{O}_{S,x}/J_x\mathcal{V}$.

7 Conclusion

In many applications (say, in the theory of dynamical systems, bifurcation theory, in economic, biology, chemistry, etc.) a stable equilibrium state describes the established conditions in the corresponding real system (see [8], [5]). When it merges with an unstable equilibrium state the system must jump to a completely different state: as the parameter is changed the equilibrium condition in the neighbourhood considered suddenly disappears. The described results allow one to investigate in detail jumps of this kind with the use of invariants of bifurcation sets and discriminants associated with deformations of a complex system.

Bibliography

1. A.G. Aleksandrov. Nonisolated hypersurface singularities. In V.I. Arnold, editor, Theory of singularities and its applications, Advances in Soviet Mathematics, volume 1, pages 211–246. Amer. Math. Soc., Providence, RI, 1990.

2. A.G. Aleksandrov. Index of vector fields and logarithmic differential forms. Funct. Anal. and Appl., 39(4):1–13, 2005.

3. A.G. Aleksandrov. Generalized Fuchsian systems of differential equations. Journal of Mathematical Sciences (N.Y.), 39(6): 689–699, 2006.

4. V.I. Arnol'd. Catastrophe theory. Znanie, Moscow, 1981.

5. D.K. Arrowsmith, C.M. Place. Ordinary differential equations. A qualitative approach with applications. Chapman and Hall, London New York, 1982.

6. N.A. Bobylëv. Topological index of extremals of multidimensional variational problems. Funct. Anal. and Appl., 20(2):8–13, 1986.

7. X. Gómez-Mont. An algebraic formula for the index of a vector field on a hypersurface with an isolated singularity. Journal of Algebraic Geometry, 7(4):731–752.

8. J.E. Marsden, M. McCracken. The Hopf bifurcation and its applications. Springer-Verlag, New York, 1976.

9. H. Poincaré. Sur les propriétés des fonctions définies par les équations aux différences partielles. Thèse inaugurale, Paris, 1879.

10. K. Saito. On the uniformization of complements of discriminant loci. In: Hyperfunctions and Linear partial differential equations, RIMS Kōkyūroku, volume 287, pages 117–137. RIMS Kyoto University, Kyoto, 1977.

11. H. Whitney. On singularities of mappings of Euclidean spaces I. Mappings of the plane into the plane. Annals of Mathematics (2), 62:374–410, 1956.

ЛОГАРИФМИЧЕСКИЕ ДИФФЕРЕНЦИАЛЬНЫЕ ФОРМЫ И КОМПЛЕКСНЫЕ ДИНАМИЧЕСКИЕ СИСТЕМЫ

А.Г. Александров[1), А.А. Кастро, мл.[2), В.А. Грузман[1)

[1) Институт проблем управления РАН,
ул. Профсоюзная, 65, Москва, 117997, Россия, e-mail: ag_aleksandrov@mail.ru
[2) Федеральный университет штата Баия,
университет, Баия, 40170-110, Бразилия, e-mail: armando@impa.br

Аннотация. Широкий класс сложных динамических систем может быть описан как эволюционный процесс, заданный векторным полем с полиномиальными, аналитическими или гладкими коэффициентами в фазовом пространстве. Такие системы исследуются методом возмущений и анализом пространств управления и поведения вместе с соответствующим бифуркационным множеством и дискриминантом. Описывается подход к изучению таких систем, основанный на методах теории логарифмических дифференциальных форм, теории деформаций и интегрируемых связностей, ассоциированных с деформациями. Такая связность может быть представлена в виде голономной системы дифференциальных уравнений фуксового типа, коэффициенты которой обладают логарифмическими полюсами вдоль бифуркационного множества или дискриминанта деформации. Кратко обсуждается и другое интересное приложение – новый метод вычисления топологического индекса комплексного векторного поля на гиперповерхности с произвольными особенностями.

Ключевые слова: логарифмические дифференциальные формы, гиперповерхность с особенностями, кручение дифференциалов, регулярные мероморфные дифференциальные формы, форма-вычет, индекс векторного поля.

УДК 511

ON SOME EXPANSIONS OF THE NUMBER $\zeta(3)$ IN CONTINUED FRACTIONS

L.A. Gutnik

Moscow State Institute of Electronics and Mathematics,
Bol'shoj Trekhsvjatitel'skii per., 1-3/12, Moscow, 109028, Russian Federation, e-mail:
gutnik@gutnik.mccme.ru

Abstract. In this note new expansions of $\zeta(3)$ in continued fractions are obtained.
Keywords: zeta-function, $\zeta(3)$, expansion in continued fractions.

Foreword

This article is a brief account of my talk given at Moscow session of China-Russia Symposium "Complex Analysis and its applications"on October 24, 2009.

Preliminaries

Given two sequences of variables

$$\{a_\nu\}_{\nu=1}^{+\infty} \text{ and } \{b_\nu\}_{\nu=0}^{+\infty} \tag{1.1}$$

we can produce the following sequences of fractions:

$$R_0(b_0) = b_0,\ R_1(b_0, a_1, b_1) = b_0 + \frac{a_1}{R_0(b_1)}, ... \tag{1.2}$$

$$R_\nu(b_0, a_1,\ b_1,\ ...,\ a_\nu,\ b_\nu) = b_0 + \frac{a_1}{R_{\nu-1}(b_1, a_2,\ b_2,\ ...,\ a_\nu,\ b_\nu)} \tag{1.3}$$

for a positive integer ν. They are called the fraction R_ν by the finite continued fraction generated by sequences (1.1). Below we use the following standard notation:

$$R_\nu = b_0 + \frac{a_1|}{|b_1} + ... + \frac{a_\nu|}{|b_\nu} \tag{1.4}$$

If all elements of sequences (1.1) are complex numbers (but not variables), all fractions (1.4) are well defined for these complex numbers and there exists the limit

$$\lim_{\nu\to\infty} R_\nu = \alpha,$$

then it is said that α has expansion in continued fraction

$$\alpha = b_0 + \frac{a_1|}{|b_1} + ... + \frac{a_\nu|}{|b_\nu} + ... \tag{1.5}$$

Partially supported by the Russian Foundation of Basic Research (RFBR) and by the National Natural Science Foundation of China (NSFC) in the framework of the bilateral project "Complex Analysis and its applications"(project No. 08-01-92208)

They are called the finite continued fractions (1.4) as convergents of the continued fraction (1.5). Let us consider further the following difference equation

$$x_{\nu+1} - b_{\nu+1}x_\nu - a_{\nu+1}x_{\nu-1} = 0, \tag{1.6}$$

with nonnegative ν. Let $\{P_\nu\}_{\nu=-1}^{+\infty}$ and $\{Q_\nu\}_{\nu=-1}^{+\infty}$ be solutions of this equation with the following initial values

$$P_{-1} = 1,\ Q_{-1} = 0,\ P_0(b_0) = b_0,\ Q_0(b_0) = 1. \tag{1.7}$$

Then easy induction shows that P_ν and Q_ν are numerator and denominator of finite continuous fraction (1.4). Since we have the equality

$$\begin{pmatrix} P_{\nu+1} \\ Q_{\nu+1} \end{pmatrix} = b_{\nu+1} \begin{pmatrix} P_\nu \\ Q_\nu \end{pmatrix} + a_{\nu+1} \begin{pmatrix} P_{\nu-1} \\ Q_{\nu-1} \end{pmatrix}, \tag{1.8}$$

it follows that

$$\Delta_{\nu+1} = det \begin{pmatrix} P_{\nu+1} & P_\nu \\ Q_{\nu+1} & Q_\nu \end{pmatrix} = -a_{\nu+1}\Delta_\nu = (-1)^\nu \prod_{k=1}^{\nu+1} a_k,$$

and therefore

$$\frac{P_{\nu+1}}{Q_{\nu+1}} - \frac{P_\nu}{Q_\nu} = (-1)^\nu \frac{\prod\limits_{k=1}^{\nu+1} a_k}{Q_\nu Q_{\nu+1}}. \tag{1.9}$$

It follows from Apéry results that the number $\alpha = \zeta(3)$ has the expansion in the continued fraction (1.5) with

$$b_0 = 0,\ a_1 = 6, b_1 = 5,\ a_{\nu+1} = -\nu^6,\ b_{\nu+1} = \tag{1.10}$$
$$34\nu^3 + 51\nu^2 + 27\nu + 5,$$

where $\nu \in \mathbb{N}$. Yu.V. Nesterenko (1996) has offered the following expansion of the number $\zeta(3)$ in continuous fraction:

$$\zeta(3) = 1 + \frac{1|}{|4} + \frac{4|}{|4} + \frac{1|}{|3} + \frac{4|}{|2}..., \tag{1.11}$$

with a_ν and b_ν given by the following equalities

$$b_0 = 1,\ a_1 = 1,\ b_1 = a_2 = b_2 = 4. \tag{1.12}$$

$$b_{4k+1} = 2k + 2,\ a_{4k+1} = k(k+1),\ b_{4k+2} = \tag{1.13}$$
$$2k + 4,\ a_{4k+2} = (k+1)(k+2)$$

for $k \in \mathbb{N}$,

$$b_{4k+3} = 2k + 3,\ a_{4k+3} = (k+1)^2,\ b_{4k+4} = \tag{1.14}$$
$$2k + 2,\ a_{4k+4} = (k+2)^2$$

for $k \in \mathbb{N}_0$.

Let $P_\nu^\wedge$ and $Q_\nu^\wedge$ be numerator and denominator of Nesterenko fractions. It is easy to prove that numerator and denominator of Nesterenko fractions with subscript $4\nu - 2$ are equal to the numerator and denominator of Apéry fraction with subscript ν, respectively.

2 The main result

The goal of present work is to give some supplements to Apéry's and Nesterenko's results. Our research is based on the results about difference systems connected with Mejer's functions; I gave a talk about these results on conference in memory of professor N.M.Korobov.

Thus, we have found the following expansions of the number $\zeta(3)$ in continuous fractions:

Theorem A. *The number $\zeta(3)$ has the following two expansions in continued fraction: the first one is*

$$2\zeta(3) = b_0^{(*1)} + \frac{a_1^{(*1)}|}{|b_1^{(*1)}} + ... + \frac{a_\nu^{(*1)}|}{|b_\nu^{(*1)}} + ..., \tag{2.1}$$

with b_ν and a_ν given by the equalities

$$b_0^{(*1)} = 3,\ a_1^{(*1)} = -81,$$

$$a_\nu^{(*1)} = -(\nu-1)^3\nu^3(4\nu^2-4\nu-3)^3$$

for $\nu \in [2, +\infty) \cap \mathbb{N}$,

$$b_\nu^{(*1)} = 4(68\nu^6 - 45\nu^4 + 12\nu^2 - 1)$$

for $\nu \in \mathbb{N}$, and the second is

$$2\zeta(3) = b_0^{(*2)} + \frac{a_1^{(*2)}|}{|b_2^{(*2)}} + ... + \frac{a_\nu^{(*2)}|}{|b_\nu^{(*2)}} + ..., \tag{2.2}$$

with b_ν and a_ν given by the equalities

$$b_0^{(*2)} = 2,\ a_1^{(*2)} = 42,$$

$$a_\nu^{(*2)} = -(\nu-1)^3\nu^3(4\nu^2-4\nu-3)((\nu+1)^3-\nu^3)((\nu-1)^3-(\nu-2)^3)$$

for $\nu \in [2, +\infty) \cap \mathbb{N}$,

$$b_\nu^{(*2)} = 2(102\nu^6 - 68\nu^4 + 21\nu^2 - 3),$$

for $\nu \in \mathbb{N}$.

As a result we specify also a way to obtain many other expansions of the number $\zeta(3)$ in continued fractions.

The next three sections contain a sketch of proof of Theorem A.

3 Auxiliary functions

Suppose that z satisfies to the following conditions:

$$|z| > 1, -3\pi/2 < \arg(z) \le \pi/2, \log(z) = \ln(|z|) + i\arg(z), \tag{3.1}$$

let δ be the differentiation $z\frac{\partial}{\partial z}$, and let α be a nonnegative integer. My first auxiliary function is a finite sum

$$f^{*\vee}_{\alpha,1}(z,\nu) := f^*_{\alpha,1}(z,\nu) := \sum_{k=0}^{\nu+\alpha}(z)^k \binom{\nu+\alpha}{k}^2 \binom{\nu+k}{\nu}^2. \tag{3.2}$$

Let us consider the rational function given by the equality

$$R(\alpha, t, \nu) = \frac{\prod_{j=1}^{\nu}(t-j)}{\prod_{j=0}^{\nu+\alpha}(t+j)}. \tag{3.3}$$

My second and fourth auxiliary function are sums of the following series

$$f^*_{\alpha,2}(z,\nu) = \sum_{t=1}^{+\infty} z^{-t}\frac{(\nu+\alpha)!^2}{\nu!^2}(R(\alpha,t,\nu))^2, \tag{3.4}$$

$$f^*_{\alpha,4}(z,\nu) = -\sum_{t=1}^{+\infty} z^{-t}\frac{(\nu+\alpha)!^2}{\nu!^2}\left(\frac{\partial}{\partial t}(R^2)\right)(\alpha,t,\nu). \tag{3.5}$$

Finally my third auxiliary function is defined as follows:

$$f^*_{\alpha,3}(z,\nu) = (\log(z))f^*_{\alpha,2}(z,\nu) + f^*_{\alpha,0,4}(z,\nu). \tag{3.6}$$

We consider also the functions $f_{\alpha,k}(z,\nu)$, $k = 1, 2, 3, 4$, related with previous functions by means of the equalities

$$f_{\alpha,k}(z,\nu) = \frac{\nu!^2}{(\nu+\alpha)!^2}(z,\nu)f^*_{\alpha,k}(z,\nu), \tag{3.7}$$

where $k = 1, 2, 3, 4$, $\nu \in \mathbb{N}_0$. Making use of the expansion of the following rational function

$$\frac{(\nu+\alpha)!^2}{(\nu!)^2}(-t)^r(R(\alpha,t,\nu))^2$$

into partial fractions relatively to t, and some simple transformations we obtain the following equality

$$\delta^r f^*_{\alpha,2+j}(z,\nu) - j(\log(z))\delta^r f^*_{\alpha,2}(z,\nu) = \tag{3.8}$$

$$\left(\sum_{i=1}^{2}(1-j+ij)\beta^{*(r)}_{\alpha,i}(z;\nu)L_{i+j}(1/z)\right) - \beta^{(r)}_{\alpha,3+j}(z;\nu),$$

where $\delta = z\frac{\partial}{\partial z}$, $j = 0, 1$, $r = 0, 1, 2, 3$, $|z| > 1$, $\alpha \in \mathbb{N}$, $s \in \mathbb{Z}$,

$$L_s(1/z) = \sum_{n=1}^{\infty} 1/(z^n n^s) \tag{3.9}$$

are polylogarithms and $\beta^{*(r)}_{\alpha,0,i}(z;\nu)$, $\beta^{*(r)}_{\alpha,0,3+j}(z;\nu)$, are polynomials of z with rational coefficients. It is clear that

$$L_s(1) = \zeta(s), \quad s > 1. \tag{3.10}$$

4 Passing to a difference system

In fact, the auxiliary functions $f^{\vee}_{\alpha,k}(z,\nu)$ are generalized hypergeometric functions, so called Mejer's functions. They satisfy the following differential equation

$$D_\alpha(z,\nu,\delta)f^{\vee}_{\alpha,k}(z,\nu)=0, \tag{4.1}$$

where $\nu\in[0,+\infty)\cap\mathbb{Z}$, $k\in\mathfrak{K}_0=\{1,\,2,\,3\}$,

$$D_\alpha(z,\nu,\delta)=z(\delta-\nu-\alpha)^2(\delta+\nu+1)^2-\delta^4. \tag{4.2}$$

is differential operator, and $\delta:=z\frac{\partial}{\partial z}$. It follows from general properties of the Mejer's functions that

$$(\delta+\nu+1)^2f_{\alpha,k}(z,\nu)=(\delta-\nu-1-\alpha)^2f_{\alpha,k}(z,\nu+1), \tag{4.3}$$

where $\nu\in[0,+\infty)\cap\mathbb{Z}$, $k\in\mathfrak{K}_0$. Since,

$$(1-1/z)^{-1}D_\alpha(z,\nu,\delta)=\delta^4-\sum_{k=1}^{4}b_{\alpha,k}\delta^{k-1},$$

we can obtain by standard considerations the differential system

$$\delta X_{\alpha,k}(z;\nu)=B_\alpha(z;\nu)X_{\alpha,k}(z;\nu), \tag{4.4}$$

where $k=1,\,2,\,3$, $|z|>1$, $\nu\in\mathbb{N}_0$,

$$B_\alpha(z;\nu)=\begin{pmatrix}0&1&0&0\\0&0&1&\\0&0&0&1\\b_{\alpha,1}(z;\nu)&b_{\alpha,2}(z;\nu)&b_{\alpha,3}(z;\nu)&b_{\alpha,4}(z;\nu)\end{pmatrix},$$

$$X_{\alpha,k}(z;\nu)=\begin{pmatrix}f^*_{\alpha,k}(z,\nu)\\\delta f^*_{\alpha,k}(z,\nu)\\\delta^2f^*_{\alpha,k}(z,\nu)\\\delta^3f^*_{\alpha,k}(z,\nu)\end{pmatrix},$$

where $k=1,\,2,\,3$, $|z|>1$. In view of (4.2),

$$D_\alpha(z,-\nu-\alpha-1,\delta)=D_\alpha(z,\nu,\delta). \tag{4.5}$$

Therefore we can put

$$X_{\alpha,k}(z;-\nu-1-\alpha)=X_{\alpha,k}(z;\nu), \tag{4.6}$$

where $\nu\in\mathbb{N}_0$, and then consider $X_{\alpha,k}(z;\nu)$ on

$$\nu\in M^{***}_\alpha=((-\infty,-1-\alpha]\cup[0,+\infty)\cap\mathbb{Z},$$

Finally, we use the equations (4.1), (4.3) and (4.4) to obtain the following difference system.

Theorem 1. *The column $X_{\alpha,k}(z;\nu)$ satisfies to the equation*

$$\nu^5X_{\alpha,k}(z;\nu-1)=A^*_\alpha(z;\nu)X_{\alpha,k}(z;\nu), \tag{4.7}$$

for $\nu \in M^*_\alpha = (-\infty, -1-\alpha] \cup [1, +\infty)) \cap \mathbb{Z}$, $k = 1, 2, 3$, $|z| > 1$; *moreover, the matrix* $A^*_\alpha(z;\nu)$ *has the following property:*

$$-\nu^5(\nu+\alpha)^5 E_4 = A^*_\alpha(z; -\nu-\alpha) A_{\alpha},^*(z;\nu), \tag{4.8}$$

where E_4 *is the* 4×4 *unit matrix,* $z \in \mathbb{C}$, $\nu \in \mathbb{C}$.

Although the matrix $A^*_\alpha(z;\nu)$ is a 4×4-matrix, its elements are huge polynomials in $\mathbb{Q}[z,\nu,\alpha]$. For example, if we put

$$\mu = \mu_\alpha(\nu) = (\nu+\alpha)(\nu+1),\ \tau = \tau_\alpha(\nu) = \nu + \frac{1+\alpha}{2}, \tag{4.9}$$

then the the matrix $A^*_\alpha(z;\nu)$ has on intersection of its first row and its first column the element

$$a^*_{\alpha,1,1}(z,\nu) = a^\vee_{\alpha,1,1}(z,\nu) + \tau a^\wedge_{\alpha,1,1}(z,\nu),$$

where

$$a^\vee_{\alpha,1,1}(z,\nu) = \frac{1}{2}(-1+2\alpha-\alpha^2-5\mu+3\alpha\mu-5\mu^2-\alpha\mu^2)+ \tag{4.10}$$

$$\frac{z}{2}(-4+12\alpha-13\alpha^2+6\alpha^3-\alpha^4)+$$

$$\frac{z}{2}\mu(-32+54\alpha-29\alpha^2+5\alpha^3-56\mu+20\alpha\mu),$$

$$a^\wedge_{\alpha,1,1}(z;\nu) = 1-\alpha+3\mu+\mu^2+ \tag{4.11}$$

$$z(4-8\alpha+5\alpha^2-\alpha^3+24\mu-22\alpha\mu+5\alpha^2\mu+16\mu^2),$$

So, the equality (4.8) was very helpful for us, when we have checked our calculations.

5 Reducing the obtained system to the difference system of the second order in the case $\alpha = 1$.

This is key point in our research, it leads to desirable results. In the case $\alpha = 1$ the situation simplifies since the above system is reducible and our problem can be reduced to the consideration of a system of the second order. To be more precise, in this case

$$\tau = \tau_1(\nu) = \nu+1,\ \mu = \mu_1(\nu) = (\nu+1)^2, \tag{5.1}$$

$$\frac{1}{z}D_\alpha(z,\nu,\delta) = (1-1/z)\delta^4 + \sum_{k=0}^{3} r_{\alpha,k+1}(\nu)\delta^k, \tag{5.2}$$

where

$$r_1(\nu) = \mu_1(\nu)^2 = (\nu+1)^4 = \tau^4,\ r_2(\nu) = 0,$$

$$r_3(\nu) = -2\mu_1(\nu) = -2(\nu+1)^2,\ r_4(\nu) = 0,$$

Let us consider the row

$$R(\nu) = (r_1(\nu),\ r_2(\nu),\ r_(\nu),\ r_4(\nu)). \tag{5.3}$$

Let E_4 be the 4×4-unit matrix, and let $C(\nu)$ be the result of replacement of 1-th row of the matrix E_4 by the row in (5.3). Let further $D(\nu)$ be the adjoint matrix to the matrix $C(\nu)$. Then

$$C(\nu)D(\nu) = \mu^2 E_4,\ C(-\nu-2) = C(\nu),\ D(-\nu-2) = D(\nu), \tag{5.4}$$

Set

$$A_1^{**}(1,\nu) = C(\nu-1)A_1^*(z,\nu)D(\nu). \tag{5.5}$$

and

$$Y_{1,k}(z;\nu) = \begin{pmatrix} y_{1,1,k}(z;\nu) \\ y_{1,2,k}(z;\nu) \\ y_{1,3,k}(z;\nu) \\ y_{1,4,k}(z;\nu) \end{pmatrix} = C(\nu)X_{1,k}(z;\nu), \tag{5.6}$$

where $k = 1,\ 2,\ 3,\ |z| > 1,\ \nu \in M_1^{***} = ((-\infty,-2] \cup [0,+\infty)) \cap \mathbb{Z}$. Then

$$Y_{1,k}(z;-\nu-2) = Y_{1,k}(z;\nu), \tag{5.7}$$

$$\mu_1(\nu)^2\nu^5 Y_{1,k}(z;\nu-1) = A_1^{**}(z,\nu)Y_{1,k}(z;\nu), \tag{5.8}$$

where $\kappa = 0,\ 1,\ k = 1,\ 2,\ 3,\ |z| > 1,\ \nu \in M_1^* = ((-\infty,-2] \cup [1,+\infty)) \cap \mathbb{Z}$. Replacing in the equality (5.8)

$$\nu \in M_1^* = ((-\infty,-2] \cup [1,+\infty)) \cap \mathbb{Z}$$

by

$$\nu := -\nu - 2 \in M_1^{**} = ((-\infty,-3] \cup [0,+\infty)) \cap \mathbb{Z},$$

and taking in account (5.7) we obtain the equality

$$\mu_1(\nu)^2(\nu+2)^5 Y_{1,k}(z;\nu+1) = -A_1^{**}(z,-\nu-2)Y_{1,k}(z;\nu), \tag{5.9}$$

where $k = 1,\ 2,\ 3,\ |z| > 1,\ \nu \in M_1^{**} = ((-\infty,-3] \cup [0,+\infty)) \cap \mathbb{Z}$. The matrix $A_1^{**}(z,\nu)$ can be represented in the form

$$A_1^{**}(z;\nu) = A_1^{**}(1;\nu) + (z-1)V_1^{**}(\nu), \tag{5.10}$$

where the matrix $V_1^{**}(\nu)$ does not depend from z. We will tend $z \in (1,+\infty)$ to 1. Therefore we must study the behavior of our auxiliary functions, when tend $z \in (1,+\infty)$ to 1. Then

$$t^r R(1,t,\nu)^2 = \frac{\prod\limits_{j=1}^{\nu}(t-j)^2}{\prod\limits_{j=0}^{\nu+1}(t+j)^2} = t^{r-4} + t^{r-5}O(1)\ (t \to +\infty) \tag{5.11}$$

$$t^r\left(\frac{\partial}{\partial t}(R^2)\right)(1,t,\nu) = t^{r-5}O(1)\ (t \to +\infty) \tag{5.12}$$

for $r = 0,\ 1,\ 2\,3,\ 4$. Therefore

$$(z-1)\delta^r f_{1,2}(z,\nu) = \tag{5.13}$$

$$\sum_{t=1}^{+\infty} z^{-t}(-t)^r(R(\alpha,t,\nu))^2 = (z-1)O(1)\ln(1-1/z) \to 0\ (z \to 1+0)$$

for $r = 0,\ 1,\ 2,\ 3,$

$$(z-1)\delta^4 f_{1,2}(z,\nu) = \tag{5.14}$$

$$\sum_{t=1}^{+\infty} z^{-t}(-t)^4(R(\alpha,t,\nu))^2 = 1 + (z-1)O(1)\ln(1-1/z) \to 1\ (z \to 1+0)$$

$$(z-1)\delta^r f_{1,4}(z,\nu) = \tag{5.15}$$

$$-\sum_{t=1}^{+\infty} z^{-t}(-t)^r \left(\frac{\partial}{\partial t}(R^2)\right)(1,t,\nu) = (z-1)O(1) \to 0 \ (z \to 1+0)$$

for $r = 0, 1, 2, 3, 4$ and

$$(z-1)\delta^r f_{1,3}(z,\nu) = (z-1)(\log(z))\delta^r f_{1,2}(z,\nu) + \tag{5.16}$$

$$(z-1)r\delta^{r-1} f_{1,2}(z,\nu) + (z-1)\delta^r f_{1,4}(z,\nu) \to 0 \ (z \to 1+0)$$

for $r = 0, 1, 2, 3, 4$. Further we have

$$y_{1,j+1,k}(z,\nu) = \delta^j f_{1,k}(z,\nu), \tag{5.17}$$

where $j = 1, 2, 3\, k = 1, 2, 3$, $|z| > 1$, $\nu \in \mathbb{N}_0$. Further we have

$$y_{1,1,k}(1,\nu) := \lim_{z\to 1+0} y_{1,1,k}(z,\nu) = \tag{5.18}$$

$-\lim\limits_{z\to 1+0}(1 - 1/z)\delta^4 f_{1,k}(z,\nu) = (k-1)(k-3)$, where $k = 1, 2, 3, , \nu \in \mathbb{N}_0$,

$$A_1^{**}(1;\nu) = \begin{pmatrix} (\nu+1)^4\nu^5 & 0 & 0 & 0 \\ a^{**}_{1,2,1}(1;\nu) & a^{**}_{1,2,2}(1;\nu) & a^{**}_{1,2,3}(1;\nu) & 0 \\ a^{**}_{1,3,1}(1;\nu) & a^{**}_{1,3,2}(1;\nu) & a^{**}1,3,3(1;\nu) & 0 \\ a^{**}_{1,4,1}(1;\nu) & a^{**}_{1,4,2}(1;\nu) & a^{**}_{1,4,3}(1;\nu) & (\nu+1)^4\nu^5 \end{pmatrix} \tag{5.19}$$

with

$$a^{**}_{1,2,1}(1;\nu) = \tag{5.20}$$

$$-\tau^2(\tau-1)(2\tau-1)(6\tau^2-4\tau+1),$$

$$a^{**}_{1,2,2}(1;\nu) = \tau^5(\tau-1)(\tau^3+2(2\tau-1)^3), \tag{5.21}$$

$$a^{**}_{1,2,3}(1;\nu) = -3\tau^4(\tau-1)(2\tau-1)^3, \tag{5.22}$$

$$a^{**}_{1,3,1}(1;\nu) = \tag{5.23}$$

$$\tau^2(\tau-1)^2(2\tau-1)(4\tau^2-3\tau+1),$$

$$a^{**}_{1,3,2}(1;\nu) = \tag{5.24}$$

$$-2\tau^5(\tau-1)^2(2\tau-1)(\tau^3-(\tau-1)^3),$$

$$a^{*0*}_{1,3,3}(1;\nu) = \tag{5.25}$$

$$\tau^4(\tau-1)^2((\tau-1)^3+2(2\tau-1)^3),$$

$$a^{**}_{1,4,1}(1;\nu) = \tag{5.26}$$

$$-\tau^2(\tau-1)^3(2\tau-1)(2\tau^2-2\tau+1),$$

$$a^{**}_{1,4,2}(1;\nu) = \tag{5.27}$$

$$\tau^5(\tau-1)^3(2\tau-1)(4\tau^2-5\tau+3).$$

$$a^{**}_{1,4,3}(1;\nu) = \tag{5.28}$$

$$-\tau^4(\tau-1)^3(2\tau-1)(6\tau^2-8\tau+3).$$

If we consider the second and third equations in the system of equations (5.8) with $k=1,\,3$ and tend $z\in(1,+\infty)$ to 1, then, in view of (5.18) and (5.19), we obtain equations

$$\mu_1(\nu)^2\nu^5\delta^i f_{1,0,k}(1,\nu-1)= \tag{5.29}$$

$$\left(\sum_{j=1}^{2} a^{**}_{1,0,i+1,j+1}(1;\nu)(\delta^j f_{1,0,k})(1,\nu)\right),$$

where $i=1,\,2$, $k=1,\,3$ and $\nu\in M_1^*=((-\infty,-2]\cup[1,+\infty))\cap\mathbb{Z}$.

Let us take

$$F=\{F(\nu)\}_{nu=-\infty}^{+\infty} \text{ and } G=\{G(\nu)\}_{nu=-\infty}^{+\infty} \tag{5.30}$$

such that

$$F(-\nu-2)=F(\nu),\ G(-\nu-2)=G(\nu),\ F(\nu)\in\mathbb{Q},\ G(\nu)\in\mathbb{Q} \tag{5.31}$$

for $\nu\in\mathbb{Z}$. Then in view of (5.7),

$$y^{**}_{F,G}(z,-\nu-2)=y^{**}_{F,G}(z,\nu)= \tag{5.32}$$

for $\kappa=0,\,1$, $k=1,\,3$ and $\nu\in M_1^{***}=((-\infty,-2]\cup[0,+\infty))\cap\mathbb{Z}$. In view of (5.29)

$$\left(\sum_{j=1}^{2} a^{***}_{F,G,j+1}(1;\nu)(\delta^j f_{1,0,k})(1,\nu)\right)= \tag{5.33}$$

$$\mu_1(\nu)^2\nu^5 y^{***}_{F,G}(z,\nu-1),$$

where $k=1,\,3$ and $\nu\in M_1^*=((-\infty,-2]\cup[1,+\infty))\cap\mathbb{Z}$. Replacing in (5.33) $\nu\in M_1^*$ by

$$\nu:=-\nu-2\in M_1^{**}=((-\infty,-3]\cup[0,+\infty))\cap\mathbb{Z},$$

and taking in account the equality (5.7) we obtain the equalities

$$\left(\sum_{j=1}^{2} a^{**\kappa*}_{F,G,j+1-\kappa}(1;-\nu-2)(\delta^{j-\kappa} f_{1,0,k})(1,\nu)\right)= \tag{5.34}$$

$$-\mu_1(\nu)^2(\nu+1)^5 y^{**}_{F,G}(z,\nu+1),$$

where $k=1,\,3$ and $\nu\in M_1^{**}=((-\infty,-3]\cup[0,+\infty))\cap\mathbb{Z}$. Set

$$\vec{w}^{(\kappa)}_{F,G,j}(\nu)=\left(\begin{array}{c} a^{***}_{F,G,j+1}(1;-\nu-2)\\ F(\nu)(2-j)+G(\nu)(j-1)\\ a^{***}_{F,G,j+1}(1;\nu-1)\end{array}\right), \tag{5.35}$$

where $j=1,2$, $\nu\in M_1^{****}=((-\infty,-3]\cup[1,+\infty))\cap\mathbb{Z}$,

$$W_{F,G}(\nu)=\left(\begin{array}{cc}\vec{w}^{(\kappa)}_{F,G,1}(\nu) & \vec{w}^{(\kappa)}_{F,G,2}(\nu)\end{array}\right)= \tag{5.36}$$

$$\left(\begin{array}{cc} a^{***}_{F,G,2}(1;-\nu-2) & a^{***}_{F,G,3}(1;-\nu-2)\\ F(\nu) & G(\nu)\\ a^{***}_{F,G,2}(1;\nu) & a^{***}_{F,G,3}(1;\nu)\end{array}\right),\ Y^{***}_k(\nu)=$$

$$\begin{pmatrix} (\delta f_{1,0,k})(1,\nu) \\ (\delta^2 f_{1,0,k})(1,\nu) \end{pmatrix},$$

$$Y^{****}_{F,G,k}(\nu) = \tag{5.37}$$

$$\begin{pmatrix} -\mu_1(\nu)^2(\nu+2)^5 y^{**}_{F,G}(z,-\nu-2) \\ y^{**}_{F,G}(z,\nu) \\ \mu_1(\nu)^2\nu^5 y^{**}_{F,G}(z,\nu-1) \end{pmatrix},$$

where $k = 1,\ 3,\ \nu \in M_1^{****} = ((-\infty,-3] \cup [1,+\infty)) \cap \mathbb{Z}$. In view of (5.33) and (5.34)

$$Y^{****}_{F,G,k}(\nu) = W_{F,G}(\nu) Y^{***}_k(\nu). \tag{5.38}$$

Let further

$$\vec{w}_{F,G,3}(\nu) = \begin{pmatrix} w_{F,G,3,1}(\nu) \\ w_{F,G,3,2}(\nu) \\ w_{F,G,3,3}(\nu) \end{pmatrix} = [\vec{w}_{F,G,1}(\nu), \vec{w}_{F,G,2}(\nu)] \tag{5.39}$$

be the vector product of $\vec{w}^{F,G,1}(\nu)$ and $\vec{w}_{F,2}(\nu)$, and let

$$\bar{w}_{F,G,3}(\nu) = (\vec{w}_{F,G,3}(\nu))^t$$

be the row conjugate to the column $\vec{w}_{F,G,3}(\nu)$. Then for scalar products

$$(\vec{w}^{(\kappa)}_{F,G,3}(\nu), \vec{w}^{(\kappa)}_{F,G,j}(\nu))$$

we have the equalities

$$\bar{w}^{(\kappa)}_{F,G,3}(\nu)\vec{w}^{\kappa}_{F,G,j}(\nu) =$$

$$(\vec{w}^{(\kappa)}_{F,G,3}(\nu), \vec{w}^{(\kappa)}_{F,G,j}(\nu)) = 0,$$

where $\kappa = 0,\ 1,\ j = 1,\ 2$ and

$$\nu \in M_1^{****} = ((-\infty,-3] \cup [1,+\infty)) \cap \mathbb{Z}.$$

Therefore

$$\bar{w}_{F,G,3}(\nu) W_{F,G}(\nu) = \begin{pmatrix} 0 & 0 \end{pmatrix}, \tag{5.40}$$

where $\nu \in M_1^{****} = ((-\infty,-3] \cup [1,+\infty)) \cap \mathbb{Z}$. In view of (5.29), (5.34) and (5.40),

$$\bar{w}(\kappa)_{F,G,3}(\nu) Y^{**\kappa***}_{F,G,k}(\nu) = \tag{5.41}$$

$$\bar{w}(\kappa)_{i,3}(\nu) W(\kappa)_i(\nu) Y^{***}_k(\nu) = 0,$$

where $k = 1,\ 3$ and $\nu \in M_1^{****} = ((-\infty,-3] \cup [1,+\infty)) \cap \mathbb{Z}$.

Thus, for given F and G we obtain a difference equation of the second order, which leads to desirable results. First, taking $F(\nu) = 1$ and $G(\nu) = 0$ for all $\nu \in \mathbb{Z}$, we then obtain the first expansion described in Theorem A. Further, taking $F(\nu) = 0$ and $G(\nu) = 1$ for all $\nu \in \mathbb{Z}$, we then obtain the second expansion from Theorem A.

Acknowledgement. I express my thanks to A.G.Aleksandrov for his support.

Bibliography

1. R.Apéry. Interpolation des fractions continues et irrationalite de certaines constantes, Bulletin de la section des sciences du C.T.H., 1981, No 3, 37 – 53;
2. Oskar Perron. Die Lehre von den Kettenbrüche. Dritte, verbesserte und erweiterte Auflage. 1954 B.G.Teubner Verlaggesellshaft. Stuttgart.
3. Yu.V. Nesterenko. A Few Remarks on $\zeta(3)$, Mathematical Notes, Vol 59, No 6, 1996, Matematicheskie Zametki, Vol 59, No 6, pp. 865 – 880 1996 (in Russian).
4. L.A.Gutnik. On linear forms with coefficients in $\mathbb{N}\zeta(1+\mathbb{N})$ (the detailed version,part 3), Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2002, 57, 1 – 33;
5. On the measure of nondiscreteness of some modules, Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2005, 32, 1 – 51.
6. On the Diophantine approximations of logarithms in cylotomic fields. Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2006, 147, 1 – 36.
7. On some systems of difference equations. Part 1. Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2006, 23, 1 – 37.
8. On some systems of difference equations. Part 2. Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2006, 49, 1 – 31.
9. On some systems of difference equations. Part 3. Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2006, 91, 1 – 52.
10. On some systems of difference equations. Part 4. Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2006, 101, 1 – 49.
11. On some systems of difference equations. Part 5. Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2006, 115, 1 – 9.
12. On some systems of difference equations. Part 6. Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2007, 16, 1 – 30.
13. On some systems of difference equations. Part 7. Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2007, 53, 1 – 40.
14. On some systems of difference equations. Part 8. Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2007, 64, 1 – 44.
15. On some systems of difference equations. Part 9. Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2007, 129, 1 – 36.
16. On some systems of difference equations. Part 10. Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2007, 131 , 1 – 33.
17. On some systems of difference equations. Part 11. Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2008, 38 , 1 – 45.

О НЕКОТОРЫХ РАЗЛОЖЕНИЯХ $\zeta(3)$ В НЕПРЕРЫВНЫЕ ДРОБИ

Л.А. Гутник

Московский государственный институт электроники и математики,

Большой Трехсвятителский пер., 1-3/12, Москва, 109028, Россия, e-mail: gutnik@gutnik.mccme.ru

Аннотация. В данной работе получены новые разложения $\zeta(3)$ в непрерывные дроби.

Ключевые слова: дзета-функция, $\zeta(3)$, разложения в непрерывные дроби.

2000 MSC 32A15, 32A20, 32A22, 32H30

UNIQUENESS THEOREMS OF MEROMORPHIC FUNCTIONS IN SEVERAL COMPLEX VARIABLES

Pei-Chu Hu[1], Chung-Chun Yang[2]

[1] Department of Mathematics, Shandong University,
Jinan 250100, Shandong, P. R. China e-mail: pchu@sdu.edu.cn
[2] Department of Mathematics, The Hong Kong University of Science & Technology
Clear Water Bay, Kowloon, Hong Kong e-mail: chungchun.yang@gmail.com

Abstract. In the survey, results on the existence, growth, uniqueness, and value distribution of meromorphic (or entire) solutions of homogeneous linear partial differential equations of the second order with polynomial coefficients that are similar or different from that of meromorphic solutions of linear ordinary differential equations have been obtained. We have characterized those entire solutions of a special partial differential equation that relate to Bessel functions and prove in general that meromorphic solutions that grow much faster than the coefficient have zero Nevanlinna's deficiency for each non-zero complex value. It's well-know result that if a nonconstant meromorphic function f on $\mathbb{C}$ and its l-th derivative $f^{(l)}$ have no zeros for some $l \geq 2$, then f is of the form $f(z) = \exp(Az + B)$ or $f(z) = (Az + B)^{-n}$ for some constants A, B. We have extended this result to meromorphic functions of several variables, by first extending the classic Tumura-Clunie theorem for meromorphic functions of one complex variable to that of meromorphic functions of several complex variables by utilizing Nevanlinna theory.

Keywords: meromorphic functions, homogeneous linear partial differential equation, holomorphic coefficients, Nevanlinna's value distribution theory.

Analytic properties or characterizations of meromorphic (or entire) solutions of some partial differential equations (or system) of the first order have been exhibited clearly by several authors (cf. [2], [13], [18], [19]). In this survey, we introduce a few results on meromorphic solutions of homogeneous linear partial differential equations of the second order in two independent complex variables

$$a_0\frac{\partial^2 u}{\partial t^2} + 2a_1\frac{\partial^2 u}{\partial t\partial z} + a_2\frac{\partial^2 u}{\partial z^2} + a_3\frac{\partial u}{\partial t} + a_4\frac{\partial u}{\partial z} + a_6 u = 0, \tag{1.1}$$

where $a_k = a_k(t, z)$ are holomorphic functions for $(t, z) \in \Sigma$, where Σ is a region on $\mathbb{C}^2$. Basic idea comes from S. N. Bernšteĭn [3], H. Lewy [17], I. G. Petrovskiĭ[20]. For more detail, see [15]. To prove these results, we used some methods in [5], [7], [11], [14], [21], [23] and [26].

First of all, we examine the following special differential equation:

$$t^2\frac{\partial^2 u}{\partial t^2} - z^2\frac{\partial^2 u}{\partial z^2} + t\frac{\partial u}{\partial t} - z\frac{\partial u}{\partial z} + t^2 u = 0. \tag{1.2}$$

The work of Chung-Chun Yang was partially supported by Natural Science Foundation of China and second author was partially supported by a UGC Grant of Hong Kong: Project 604106.

Theorem 1.1 *The differential equation* (1.2) *has an entire solution* $f(t,z)$ *on* $\mathbb{C}^2$ *if and only if* f *is an entire function expressed by the series*

$$f(t,z)=\sum_{n=0}^{\infty} n! c_n J_n(t) z^n \tag{1.3}$$

such that

$$\limsup_{n\to\infty} |c_n|^{1/n}=0, \tag{1.4}$$

where $J_n(t)$ *is the first kind of Bessel's function of order* n*. Moreover, the order* $\mathrm{ord}(f)$ *of the entire function* f *satisfies*

$$\rho\le \mathrm{ord}(f)\le \max\{1,\rho\},$$

where

$$\rho=\limsup_{n\to\infty}\frac{2\log n}{\log|c_n|^{-1/n}}. \tag{1.5}$$

By definition, the order of f is defined by

$$\mathrm{ord}(f)=\limsup_{r\to\infty}\frac{\log^+\log^+ M(r,f)}{\log r},$$

where

$$\log^+ x=\begin{cases}\log x, & \text{if } x\ge 1;\\ 0, & \text{if } x<1,\end{cases}$$

and

$$M(r,f)=\max_{|t|\le r,|z|\le r}|f(t,z)|.$$

G. Valiron [25] showed that each transcendental entire solution of a homogeneous linear ordinary differential equation with polynomial coefficients is of finite positive order. However, Theorem 1.1 shows that Valiron's theorem is not true for general partial differential equations. Here we exhibit another example that the following equation

$$t^2\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial z^2}+t\frac{\partial u}{\partial t}=0$$

has an entire solution $\exp(te^z)$ of infinite order.

If $0<\lambda=\mathrm{ord}(f)<\infty$, we define the type of f by

$$\mathrm{typ}(f)=\limsup_{r\to\infty}\frac{\log^+ M(r,f)}{r^\lambda}.$$

For the type of entire solutions of the equation (1.2), we have an analogue of Lindelöf-Pringsheim theorem, its proof is essentially the same as that of the determining of the type for Taylor series of entire functions of one complex variable.

Theorem 1.2 *If* $f(t,z)$ *is an entire solution of* (1.2) *defined by* (1.3) *and* (1.4) *such that* $1<\lambda=\mathrm{ord}(f)<\infty$*, then the type* $\sigma=\mathrm{typ}(f)$ *satisfies*

$$e\lambda\sigma=2^{-\lambda/2}\limsup_{n\to\infty}2n|c_n|^{\lambda/(2n)}.$$

Brosch [4] proved that if two nonconstant meromorphic functions f and g on $\mathbb{C}$ share three distinct values c_1, c_2, c_3 counting multiplicities, and if f is a solution of the differential equation

$$\left(\frac{dw}{dz}\right)^n = \sum_{j=0}^{2n} b_j(z)w^j := P(z,w)$$

such that $b_0, b_1, \cdots, b_{2n}$ ($b_{2n} \not\equiv 0$) are small functions of f (grow slower than f), furthermore if $P(z, c_i) \not\equiv 0$ for $i = 1, 2, 3$, then $f = g$. To state a generalization of Brosch's result to PDE, we abbreviate

$$u_t = \frac{\partial u}{\partial t}, \; u_{tz} = \frac{\partial^2 u}{\partial t \partial z}, \; u_{tt} = \frac{\partial^2 u}{\partial t^2},$$

and so on, and set

$$Du = a_0 u_t^2 + 2a_1 u_t u_z + a_2 u_z^2,$$

$$Lu = a_0 u_{tt} + 2a_1 u_{tz} + a_2 u_{zz} + a_3 u_t + a_4 u_z.$$

We make the following assumption:

(A) All coefficients a_i in (1.1) are polynomials and when $a_6 = 0$ there are no nonconstant polynomials u satisfying the system

$$\begin{cases} Du = 0, \\ Lu = 0. \end{cases}$$

For technical reason, here we study only meromorphic functions of finite orders. The order of a meromorphic function of several variables may be defined by using its Nevanlinna's characteristic function (cf. [12], [22]).

Theorem 1.3 *Assume that the assumption* (A) *holds. Let $f(t,z)$ be a nonconstant meromorphic solution of* (1.1) *such that* $\mathrm{ord}(f) < \infty$ *and let g be a nonconstant meromorphic function of finite order on $\mathbb{C}^2$. If f and g share 0, 1, ∞ counting multiplicity, one of the following five cases is occurred:*

(a) $g = f$;

(b) $gf = 1$;

(c) $a_6 = 0$, $gf = f + g$;

(d) $a_6 = 0$, *and there exist a constant $b \notin \{0, 1\}$ and a polynomial β such that*

$$f = \frac{1}{b-1}\left(e^{\beta} - 1\right), \; g = \frac{b}{b-1}\left(1 - e^{-\beta}\right);$$

(e) $a_6 \neq 0$, $f^2 g^2 = 3fg - f - g$.

When $a_6 \neq 0$, the case (b) may happen. For example, we consider the differential equation

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial z^2} - \frac{\partial u}{\partial t} - u = 0, \tag{1.6}$$

which has an entire solution of order 1

$$f(t, z) = e^{t+z}.$$

Let's compare f with the following entire function of order 1

$$g(t, z) = e^{-t-z}.$$

Obviously, f and g share 0, 1, -1, ∞ counting multiplicity, but $g \neq f$, $gf = 1$. Now the differential equation

$$Lu + Du + a_6 = 0$$

has a nonconstant polynomial solution

$$u(t, z) = t + z.$$

The condition (A) is meaningful. For example, Theorem 1.1 shows that the differential equation (1.2) has a lot of entire solutions of finite orders. Obviously, the condition (A) associated to the differential equation (2) holds, and hence we can obtain the fact:

Corollary 1.4 *Let $f(t, z)$ be a nonconstant meromorphic solution of* (1.2) *such that* $\mathrm{ord}(f) < \infty$ *and let g be a nonconstant meromorphic function of finite order on $\mathbb{C}^2$. If f and g share* 0, 1, ∞ *counting multiplicity, then we have either $g = f$ or $gf = 1$ or $f^2 g^2 = 3fg - f - g$.*

The case (b) in Theorem 1.3 may really happen for $a_6 = 0$. For example, we consider the differential equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial u}{\partial z} = 0, \tag{1.7}$$

which has an entire solution $f(t, z) = e^{t+z}$ of order 1 such that the assumption (A) holds obviously. The entire solution f and the function $g = e^{-t-z}$ share 0, 1, ∞ counting multiplicity, and satisfy $gf = 1$, that is, the case (b) in Theorem 1.3 happens for the case $a_6 = 0$.

For a real number x, let $[x]$ denote the maximal integer $\leq x$. We give the following result that is an analogue of Anastassiadis's theorem [1] on uniqueness of entire functions of one variable.

Theorem 1.5 *Let $f(t, z)$ and $g(t, z)$ be transcendental entire solutions of* (1.2) *such that* $\mathrm{ord}(f) < \infty$, $\mathrm{ord}(g) < \infty$, *and*

$$\frac{\partial^{2j} f}{\partial t^j \partial z^j}(0, 0) = \frac{\partial^{2j} g}{\partial t^j \partial z^j}(0, 0), \; j = 0, 1, ..., q,$$

where

$$q = \max\{[\mathrm{ord}(f)], [\mathrm{ord}(g)]\}.$$

If there exists a complex number a with $(a, f(0, 0)) \neq (0, 0)$ such that f and g share a counting multiplicity, then we have $f = g$.

Theorem 1.3 shows that when $a_6 = 0$, global solutions of the equation (1.1) can be quite complicated, however, when $a_6 \neq 0$, these solutions have normal properties. Next result also supports this view. Theorem 1.6 extends a theorem (cf. Theorem 5.8 of [10]) on meromorphic solutions of linear ordinary differential equations.

Theorem 1.6 *Assume that all a_k in* (1.1) *are entire functions on $\mathbb{C}^2$ which grow slower than a meromorphic solution of equations* (1.1) *on $\mathbb{C}^2$. If $a_6 \not\equiv 0$, then the deficiency of the solution for each non-zero complex number is zero.*

For example, the telegraph equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial z^2} + 2\alpha \frac{\partial u}{\partial t} + \alpha^2 u = 0$$

has entire solutions

$$u(t,z) = e^{-\alpha t}\{f(z+ct) + g(z-ct)\},$$

where f and g are entire functions on $\mathbb{C}$. If $\alpha \neq 0$, Theorem 1.6 shows that the deficiency of a non-constant $u(t,z)$ for each non-zero complex number a is zero, which means that the equation

$$f(z+ct) + g(z-ct) - ae^{\alpha t} = 0$$

has zeros.

Let $\mathbb{Z}_+$ denote the set of non-negative integers. For $z = (z_1, ..., z_m) \in \mathbb{C}^m$, $\mathbf{i} = (i_1, ..., i_m) \in \mathbb{Z}_+^m$, we write

$$\partial_{z_k} = \frac{\partial}{\partial z_k},\ k = 1, ..., m;\ \partial^{\mathbf{i}} = \partial_z^{\mathbf{i}} = \partial_{z_1}^{i_1} \cdots \partial_{z_m}^{i_m};\ |\mathbf{i}| = i_1 + \cdots + i_m.$$

We have interesting in the following problem:

Conjecture 1.7 *If f is a meromorphic function in $\mathbb{C}^m$ such that f and $\partial^{\mathbf{l}} f$ have no zeros for some $\mathbf{l} = (l_1, ..., l_m) \in \mathbb{Z}_+^m$ with $l_k \geq 2$ $(1 \leq k \leq m)$ and such that the set of poles of f is algebraic, then there exists a partition*

$$\{1, ..., m\} = I_0 \cup I_1 \cup \cdots \cup I_k$$

such that $I_i \cap I_j = \emptyset$ $(i \neq j)$, and

$$f(z_1, ..., z_m) = \exp\left(\sum_{i \in I_0} A_i z_i + B_0\right) \prod_{j=1}^{k} \left(\sum_{i \in I_j} A_i z_i + B_j\right)^{-n_j},$$

where A_i, B_j are constants with $A_i \neq 0$, and n_j are positive integers.

This is open if $m > 1$. For detail discussion, see [16]. When $m = 1$, the conclusion of Conjecture 1.7 was obtained by Tumura [24], and Hayman [8] gave a proof for the case $l = l_m = 2$. Later, as a correction of the gap in Tumura's proof, Clunie [6] gave a valid proof of the assertion for any $l > 1$.

Let f be a meromorphic function in $\mathbb{C}^m$ which we shall assume to be not constant. We shall be concerned largely with meromorphic functions h which are polynomials in f and the partial derivatives of f with coefficients a of the form

$$\| \quad T(r,a) = o(T(r,f)), \tag{1.8}$$

where $T(r,f)$ is the *Nevanlinna's characteristic function* of f, and where the symbol "$\|$"means that the relation holds outside a set of r of finite linear measure. Such functions h will be called *differential polynomials* in f. To study Conjecture 1.7, the following result will play a crucial role.

Theorem 1.8 *Suppose that f is meromorphic and not constant in $\mathbb{C}^m$, that*

$$g = f^n + P_{n-1}(f), \tag{1.9}$$

where $P_{n-1}(f)$ is a differential polynomial of degree at most $n-1$ in f, and that

$$\| \quad N(r,f) + N\left(r, \frac{1}{g}\right) = o(T(r,f)),$$

where $N(r,f)$ is the Nevanlinna's valence function of f for poles. Then

$$g = \left(f + \frac{a}{n}\right)^n,$$

where a is a meromorphic function of the form (1.8) in $\mathbb{C}^m$ determined by the terms of degree $n-1$ in $P_{n-1}(f)$ and by g.

When $m = 1$, Theorem 1.8 is due to Hayman ([9], Theorem 3.9, p.69). By using Theorem 1.8, we can give a proof of Conjecture 1.7, under a condition on non-vanishing of the partial derivatives of order > 1 that differs from the one posed in the conjecture, as follows:

Theorem 1.9 *If f is a meromorphic function in $\mathbb{C}^m$ such that f, $\partial_{z_1}^{l_1} f$, ..., $\partial_{z_m}^{l_m} f$ have no zeros for some $l_k \geq 2$ $(1 \leq k \leq m)$ and such that the set of poles of f is algebraic, then there exists a partition*

$$\{1, ..., m\} = I_0 \cup I_1 \cup \cdots \cup I_k$$

such that $I_i \cap I_j = \emptyset$ $(i \neq j)$, and

$$f(z_1, ..., z_m) = \exp\left(\sum_{i \in I_0} A_i z_i + B_0\right) \prod_{j=1}^{k} \left(\sum_{i \in I_j} A_i z_i + B_j\right)^{-n_j},$$

where A_i, B_j are constants with $A_i \neq 0$, and n_j are positive integers.

In particular, if f is entire, the function f in Theorem 1.9 has only an exponential form

$$f(z_1, ..., z_m) = \exp\left(A_1 z_1 + \cdots + A_m z_m + B_0\right).$$

We shall utilize the methods developed in [9], [12] and [13] and generalized Clunie lemma to prove the main results.

Bibliography

1. J. Anastassiadis. Recherches algébriques sur le théorème de Picard-Montel, Exprosés sur la théorie des fonctions XVIII, Hermann, Paris, 1959.

2. C.A. Berenstein, B.Q. Li. On certain first-order partial differential equations in $\mathbb{C}^n$, Harmonic Analysis, Signal Processing, and Complexity, 29-36, Progr. Math., 238, Birkhäuser Boston, 2005.

3. S.N. Bernšteĭn. Sur la nature analytique des solutions de certaines équations aux dériveés parttielles du second order, C. R. Acad. Sci. Paris 137 (1903), 778-781.

4. G. Brosch. Eindeutigkeitssätze für meromorphe Funktionen, Thesis, Technical University of Aachen, 1989.

5. W.D. Brownawell. On the factorization of partial differential equations, Can. J. Math. Vol. XXXIX (1987), No. 4, 825-834.

6. J. Clunie. On integral and meromorphic functions, J. London Math. Soc. 37(1962), 17-27.

7. Ph. Griffiths, J. King. Nevanlinna theory and holomorphic mappings between algebraic varieties, Acta Math. 130(1973), 145–220.

8. W.K. Hayman. Picard values of meromorphic functions and their derivatives, Ann. of Math. 70(1959), 9-42.

9. W.K. Hayman. Meromorphic functions, Oxford University Press, 1964.

10. Y.Z. He, X.Z. Xiao. Algebroid functions and ordinary differential equations (Chinese), Science Press, Beijing, 1988.

11. E.W. Hobson. Spherical and Ellipsoidal Harmonics, Cambridge, 1931.

12. P.C. Hu, P. Li. C.C. Yang. Unicity of meromorphic mappings, Kluwer Academic Publishers, 2003.

13. P.C. Hu, C.C. Yang. Malmquist type theorem and factorization of meromorphic solutions of partial differential equations, Complex Variables 27 (1995), 269-285.

14. P.C. Hu, C.C. Yang. Value distribution theory related to number theory, Birkhäuser, 2006.

15. P.C. Hu, C.C. Yang. Global solutions of homogeneous linear partial differential equations of the second order, a talk at Joint Sino-Russia Symposium "Complex Analysis and its Applications Belgorod State University, Russia, 2009; to appear in Michigan Math. J. (2009).

16. P.C. Hu, C.C. Yang. Tumura-Clunie theorem in several complex variables, a talk at Joint Sino-Russia Symposium "Complex Analysis and its Applications Belgorod State University, Russia, 2009.

17. H. Lewy. Neuer Beweis des analytischen Charakters der Losungen elliptische Differentialgleichungen, Math. Ann 101(1929),609-619.

18. B.Q. Li. Entire solutions of certain partial differential equations and factorization of partial derivatives, Trans. Amer. Math. Soc. 357 (2005), no. 8, 3169-3177 (electronic).

19. B.Q. Li, E.G. Saleeby. Entire solutions of first-order partial differential equations, Complex Variables 48 (2003), No. 8, 657-661.

20. I.G. Petrovskiĭ. Dokl. Akad. Nauk SSSR 17(1937), 343-346; Mat. Sb. (N.S.) 5(47) (1939), 3-70.

21. N. Steinmetz. Über die faktorisierbaren Lösungen gewöhnlicher Differentielgleichungen, Math. Z. 170 (1980), 169-180.

22. W. Stoll. Holomorphic functions of finite orders in several complex variables, Conference Board of the Mathematical Science, Regional Conference Series in Mathematics 21, AMS, 1974.

23. W. Stoll. Value distribution on parabolic spaces, Lecture Notes in Math. 600(1977), Springer-Verlag.

24. Y. Tumura. On the extensions of Borel's theorem and Saxer-Csillag's theorem, Proc. Phys. Math. Soc. Japan (3), 19(1937), 29-35.

25. G. Valiron. Lectures on the general theory of integral functions, Toulouse: Édouard privat, 1923.

26. Z.X. Wang, D.R. Guo. Special functions, World Scientific, 1989.

ТЕОРЕМЫ ЕДИНСТВЕННОСТИ МЕРОМОРФНЫХ ФУНКЦИЙ НЕСКОЛЬКИХ КОМПЛЕКСНЫХ ПЕРЕМЕННЫХ

Пей Чу Ху[1], **Чунг Чун Янг**[2]

[1] Шаньдунский университет,

Жиньян 250100, Шаньдун, П. Р. Китай e-mail: pchu@sdu.edu.cn

[2] Гонконгский университет науки и технологии

Клеар Уотер Бэй, Ковлун, Гонконг e-mail: chungchun.yang@gmail.com

Аннотация. В работе исследуются вопросы существования, единственности и распределния значений мероморфных (или целых) решений линейных дифференциальных уравнений в частных производных второго порядка с полиномиальными коэффициентами.

Ключевые слова: мероморфные функции, однородные линейные дифференциальные уравнения в частных производных, теория Неванлинны распределения ценности.

2000 MSC 32A15, 32A20, 32A22, 32H30

NORMAL FAMILY AND THE SHARED POLYNOMIALS OF MEROMORPHIC FUNCTIONS

Feng Lü[1)], Jun-Feng Xu[2)]

[1)]Department of Mathematics,China University of Petroleum,
Dongying, Shandong 257061, China, e-mail: xujunf@gmail.com
[2)]Department of Mathematics, Wuyi University,
Jiangmen, Guangdong 529020, China, e-mail: lvfeng@mail.sdu.edu.cn

Abstract. In the paper, we study the uniqueness and the shared fixed-points of meromorphic functions and prove two main theorems which improve the results of Fang and Fang and Qiu.

Keywords: meromorphic functions, fixed-points, holomorphic coefficients, shared polynomials.

1 Introduction and main results

Schwick [8] was the first to draw a connection between values shared by functions in $\mathcal{F}$ (and their derivatives) and the normality of the family $\mathcal{F}$. Specifically, he showed that if there exist three distinct complex numbers a_1, a_2, a_3 such that f and f' share a_j $(j = 1, 2, 3)$ IM in D for each $f \in \mathcal{F}$, then $\mathcal{F}$ is normal in D.

In 2006, Wang and Yi [9]proved a uniqueness theorem for entire functions that share a polynomial with their derivatives, as follows

Theorem A. *Let f be a nonconstant entire function, let $Q(z)$ be a polynomial of degree $q \geq 1$, and let $k > q$ be an integer. If f and f' share $Q(z)$ CM, and if $f^{(k)}(z) - Q(z) = 0$ whenever $f(z) - Q(z) = 0$, then $f = f'$.*

According to BlochŠÏs principle, numerous normality criteria have been obtained by starting from Picard type theorems. On the other hand, by Nevanlinna's famous five point theorem and Montel's theorem, it is interesting to establish normality criteria by using conditions known from a sharing values theorem.

In this note, we obtain the following normal family related to Theorem A.

Theorem 1.1 *Let $\mathcal{F}$ be a family of holomorphic functions in a domain D; let $Q(z)$ be a polynomial of degree $q \geq 1$, and let $k \geq 2q + 1$ be an integer. If, for each $f \in \mathcal{F}$, we have*

$$f(z) = Q(z) \rightleftharpoons f'(z) = Q(z) \Rightarrow f^{(k)} = Q(z),$$

then $\mathcal{F}$ is normal in D.

In order to prove theorem 1.1, we need the following results, which are interesting in their own rights.

Proposition 1. *Let $\mathcal{F}$ be a family of holomorphic functions in a domain D; let $h(z)$ be a polynomial of degree $q \geq 1$; let $k > q$ be an integer. If, for each $f \in \mathcal{F}$, we have $h(z) = 0 \Rightarrow f(z) = 0$ and $f(z) = 0 \rightleftharpoons f'(z) = h(z) \Rightarrow |f^{(k)}(z)| \leq \overline{M}$, where $\overline{M}$ is a positive number, then $\mathcal{F}$ is normal in D.*

The author was supported by the NSF of China (10771121), the NSF of Guangdong Province (9452902001003278) and Excellent Young Fund of Department of Education of Guangdong (LYM08097).

Proposition 2. *Let $\mathcal{F}$ be a family of holomorphic functions in a domain D; let $Q(z)$ be a polynomial of degree $q \geq 1$; let $k \geq 2q+1$ be an integer. If, for each $f \in \mathcal{F}$, we have $Q(z) - Q'(z) = 0 \Rightarrow f(z) \neq 0$ and $f(z) = 0 \rightleftharpoons f'(z) = Q(z) - Q'(z) \Rightarrow f^{(k)}(z) = Q(z)$, then $\mathcal{F}$ is normal in D.*

2 Some Lemmas

Lemma 2.1 *[9] Let $\mathcal{F}$ be a family of functions meromorphic in a domain D, all of whose zeros have multiplicity at least k, and suppose that there exists $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$, if $\mathcal{F}$ is not normal at $z_0 \in D$, then for each $0 \leq \alpha \leq k$ there exist,*

(a) points $z_n \in D$, $z_n \to z_0$;

(b) functions $f_n \in \mathcal{F}$, and

(c) positive number $\rho_n \to 0$ such that $\rho_n^{-\alpha} f_n(z_n + \rho_n\zeta) = g_n(\zeta) \to g(\zeta)$ locally uniformly, where g is a nonconstant meromorphic function in C, all of whose zeros have multiplicity at least k, such that $g^\sharp(\zeta) \leq g^\sharp(0) = kA+1$. In particular, if $\mathcal{F}$ is a family of holomorphic functions, then $\rho(g) \leq 1$.

Lemma 2.2 *[2] Let g be a nonconstant entire function with $\rho(g) \leq 1$; let $k \geq 2$ be a positive integer; and let a be a nonzero finite value. If $g(z) = 0 \Rightarrow g'(z) = a$, and $g'(z) = a \Rightarrow g^{(k)}(z) = 0$, then $g(z) = a(z - z_0)$, where z_0 is a constant.*

Lemma 2.3 *[2] Let $\mathcal{F}$ be a family of holomorphic functions in a domain D; let $k \geq 2$ be a positive integer; and let α be a function holomorphic in D, such that $\alpha(z) \neq 0$ for $z \in D$. If for every $f \in \mathcal{F}$, $f(z) = 0 \Rightarrow f'(z) = \alpha(z)$ and $f'(z) = \alpha(z) \Rightarrow |f^{(k)}(z)| \leq h$, where h is a positive number, then $\mathcal{F}$ is normal in D.*

In order to prove theorem 1.1, we need some definitions.

Let $\Delta = \{z : |z| < r_0\}$, let $Q(z)$ be a polynomial of degree $q \geq 1$ and $R(z) = Q(z) - Q'(z) = z^m P(z)$, $P(z) \neq 0$, when $z \in \Delta$. Define that $Q_a(z) = Q(z+a)$, where a is a constant, then

$$R_a(z) = Q_a(z) - Q'_a(z) = (z+a)^m P_a(z).$$

Define $\lambda_a = \frac{f' - R_a}{f}$ and $\lambda_a(0) \neq 0$, where f is holomorphic function in Δ. Thus we get $f' = \lambda_a f + R_a = \lambda_{a1} f + \mu_{a1}$. By mathematic induction we get $f^{(k)} = \lambda_{ak} f + \mu_{ak}$ $(k \geq q+2)$, *where*

$$\mu_{ak} = R_a\{\lambda_a^{k-1} + P_{k-2}[\lambda_a]\} + R'_a\{\lambda_a^{k-2} + P_{k-3}[\lambda_a]\} + ... + R_a^{(q)}\{\lambda_a^{k-(q+1)} + P_{k-(q+2)}[\lambda_a]\} \quad (2.1)$$

and $P_{k-2}[\lambda_a], \ldots, P_{k-(q+2)}[\lambda_a]$ are differential polynomial in λ_a with degree at most $k-2, \ldots, k-(q+2)$ respectively. Let $\mu_{ak}(0) - Q_a(0) \neq 0$. Define $\psi_a(0) \neq 0$ where

$$\psi_a = \frac{R_a f^{(k)} - Q_a f'}{f}. \quad (2.2)$$

Define $\varphi_a(0) \neq 0$ where

$$\varphi_a = -[1 + (\frac{1}{\psi_a})' Q_a + \frac{1}{\psi_a} Q'_a] R_a - \frac{1}{\psi_a} Q_a R'_a. \quad (2.3)$$

Lemma 2.4 *Let $f(z)$ be analytic in the disc $\Delta = \{z : |z| < r_0\}$; let a be a complex number such that $|a| < r_0$; let $k \geq q+2$ be a positive integer. If Q_a, R_a, λ_a, μ_{ak}, ψ_a and φ_a are defined as above; if $f(0) \neq 0$, $|f' - R_a|_{z=0} \neq 0$, $R_a = 0 \Rightarrow f(z) \neq 0$ and*

$$f(z) = 0 \rightleftharpoons f'(z) = R_a \Rightarrow f^{(k)}(z) = Q_a,$$

then

$$T(r,f) \leq LD[r,f] + M\log|\frac{f' - R_a}{\psi_a^9 \varphi_a(\mu_{ak} - Q_a)f}|_{z=0} + \log|f(0)|, \tag{2.4}$$

where

$$\begin{aligned} LD[r,f] = {} & M_1[m(r,\frac{f'}{f}) + m(r,\frac{f^{(k)}}{f}) + m(r,\frac{f^{(k)}}{f'}) + m(r,\frac{f'' - R_a'}{f' - R_a}) + m(r,\frac{f^{(k)}}{f' - R_a})] \\ & + M_1 m(r,\frac{f^{(k+1)}}{f' - R_a}) + M_2[m(r,\frac{\psi_a'}{\psi_a}) + m(r,\frac{\lambda_a'}{\lambda_a}) + ... + m(r,\frac{\lambda_a^{(k-2)}}{\lambda_a})] \\ & + M_3[m(r,R_a) + m(r,R_a') + \ldots + m(r,R_a^{(q)}) + m(r,Q_a) + m(r,Q_a') + \log 2], \end{aligned}$$

and M_1, M_2, M_3 are positive numbers.

Lemma 2.5 *[1] Let $U(r)$ be a nonnegative, increasing function on an interval $[R_1, R_2](0 < R_1 < R_2 < +\infty)$; let a, b be two positive constants satisfying $b > (a+2)^2$; and let*

$$U(r) < a\{\log^+ U(\rho) + \log\frac{\rho}{\rho - r}\} + b$$

whenever $R_1 < r < \rho < R_2$. Then, for $R_1 < r < R_2$,

$$U(r) < 2a\log\frac{R_2}{R_2 - r} + 2b.$$

Lemma 2.6 *[1] Let $g(z)$ be a transcendental entire function. Then*

$$\limsup_{|z|\to\infty} |z| g^{\sharp}(z) = +\infty.$$

3 Proof of Proposition 1

Let $z_0 \in D$. If $h(z_0) \neq 0$, by Lemma 2.3, $\mathfrak{F}$ is normal at z_0. Now suppose that $h(z_0) = 0$. Without loss of generality, we may assume that $z_0 = 0$, $\Delta = \{z : |z| < \delta\} \in D$ and $h(z) = z^m b(z)$, where $b(0) = 1$ and $b(z) \neq 0 \ \ (z \in \Delta)$. We shall prove that $\mathfrak{F}$ is normal at $z = 0$.

Let $\mathfrak{F}_1 = \{F = \frac{f}{z^m} : f \in \mathfrak{F}\}$. We know that if $\mathfrak{F}_1$ is normal at $z = 0$, then $\mathfrak{F}$ is normal at $z = 0$. Thus we only need to prove $\mathfrak{F}_1$ is normal at $z = 0$.

For each $f \in \mathfrak{F}$, from $h(z) = 0 \Rightarrow f(z) = 0$, we get $z = 0$ is a zero of f. Thus we have

$$f(z) = a_n z^n + a_{n+1} z^{n+1} + ... \quad (a_n \neq 0) \ (n \geq 1),$$

and

$$f'(z) - h(z) = n a_n z^{n-1} + (n+1) a_{n+1} z^n + ... - (z^m + ...) \ \ .$$

By the assumption $f(z) = 0 \rightleftharpoons f'(z) = h(z)$, we get

$$f = \frac{1}{m+1} z^{m+1} + a_{m+2} z^{m+2} + \dots \tag{3.1}$$

Hence we get $\mathcal{F}_1$ is a family of holomorphic functions in Δ. Next we prove $\forall\ F = \frac{f}{z^m} \in \mathcal{F}_1$, $F = 0 \Rightarrow |F'| \leq M$, where $M = \max_{z\in\Delta} |b(z)| \geq 1$.

Suppose that $F(a_0) = 0$, then $f(a_0) = 0$.

If $a_0 \neq 0$, we get $F'(a_0) = \frac{f'(a_0)}{a_0^m} - \frac{mf(a_0)}{a_0^{m+1}} = b(a_0)$.

If $a_0 = 0$, we get $F'(a_0) = b(a_0) - \frac{m}{m+1} = 1 - \frac{m}{m+1} = \frac{1}{m+1}$. Thus we get $F = 0 \Rightarrow |F'| \leq M$.

Now we prove that $\mathcal{F}_1$ is normal at $z = 0$. Suppose on the contrary that $\mathcal{F}_1$ is not normal at $z = 0$, then by Lemma 2.1, we can find $z_n \to 0$, $\rho_n \to 0$ and $f_n \in \mathcal{F}$ such that

$$g_n(\zeta) = \rho_n^{-1} \frac{f_n(z_n + \rho_n\zeta)}{(z_n + \rho_n\zeta)^m} \to g(\zeta) \tag{3.2}$$

locally uniformly on C, where g is a nonconstant entire function such that $g^\sharp(\zeta) \leq g^\sharp(0) = M+1$. In particular $\rho(g) \leq 1$. Without loss of generality, we assume that $\lim\limits_{n\to\infty} \frac{z_n}{\rho_n} = c \in C$. In the following we consider two cases.

Case 1: $c = \infty$. Then $z_n \neq 0$ and $\frac{\rho_n}{z_n} \to 0$ as $n \to \infty$. Set $h_n(\zeta) = \rho_n^{-1} \frac{f_n(z_n+\rho_n\zeta)}{z_n^m}$. Then by (3.2), we get

$$h_n(\zeta) = \rho_n^{-1} \frac{f_n(z_n + \rho_n\zeta)}{(z_n + \rho_n\zeta)^m} (1 + \frac{\rho_n}{z_n}\zeta)^m \to g(\zeta). \tag{3.3}$$

We claim:

$$g(\zeta) = 0 \Rightarrow g'(\zeta) = 1 \text{ and } g'(\zeta) = 1 \Rightarrow g^{(k)}(\zeta) = 0.$$

Suppose that $g(\zeta_0) = 0$, then by Hurwitz's Theorem, there exist ζ_n, $\zeta_n \to \zeta_0$, such that (for n sufficiently large)

$$h_n(\zeta_n) = \rho_n^{-1} \frac{f_n(z_n + \rho_n\zeta_n)}{z_n^m} = 0.$$

Thus $f_n(z_n + \rho_n\zeta_n) = 0$, by the assumption we have $f_n'(z_n + \rho_n\zeta_n) = (z_n + \rho_n\zeta_n)^m b(z_n + \rho_n\zeta_n)$, then we derive that

$$g'(\zeta_0) = \lim_{n\to\infty} \frac{f_n'(z_n + \rho_n\zeta_n)}{z_n^m} = \lim_{n\to\infty} b(z_n + \rho_n\zeta_n)(1 + \frac{\rho_n}{z_n}\zeta_n)^m = b(0) = 1.$$

Thus $g(\zeta) = 0 \Rightarrow g'(\zeta) = 1$. Next we prove $g'(\zeta) = 1 \Rightarrow g^{(k)}(\zeta) = 0$. By (3.3) we know

$$\frac{f_n'(z_n + \rho_n\zeta)}{(z_n + \rho_n\zeta)^m b(z_n + \rho_n\zeta)} = \frac{f_n'(z_n + \rho_n\zeta)}{z_n^m (1 + \frac{\rho_n}{z_n}\zeta)^m b(z_n + \rho_n\zeta)} \to g'(\zeta)$$

We suppose that $g'(\zeta_0) = 1$, obviously $g' \not\equiv 1$, for otherwise $g^\sharp(0) \leq g'(0) = 1 < M+1$, which is a contradiction. Hence by Hurwitz's Theorem, there exist ζ_n, $\zeta_n \to \zeta_0$, such that (for n sufficiently large)

$$\frac{f_n'(z_n + \rho_n\zeta_n)}{(z_n + \rho_n\zeta_n)^m b(z_n + \rho_n\zeta_n)} = 1,$$

Thus $f_n'(z_n+\rho_n\zeta_n)=h(z_n+\rho_n\zeta_n)$, by the assumption we get $|f_n^{(k)}(z_n+\rho_n\zeta_n)|\le\overline{M}$. Then

$$|g^{(k)}(\zeta_0)|=\lim_{n\to\infty}|\frac{\rho_n^{k-1}}{z_n^m}f_n^{(k)}(z_n+\rho_n\zeta_n)|\le\lim_{n\to\infty}|\frac{\rho_n^{k-1}}{z_n^m}|\overline{M}=0.$$

Thus we prove the Claim. By Lemma 2.2, we get $g=\zeta-b$, where b is a constant. Thus we have $g^\sharp(0)\le 1<M+1$, which is a contradiction.

Case 2: $c\neq\infty$. We set

$$G_n(\zeta)=\frac{f_n(\rho_n\zeta)}{\rho_n^{m+1}}. \tag{3.4}$$

Then

$$G_n(\zeta)=\rho_n^{-1}\frac{f_n(z_n+\rho_n(\zeta-\frac{z_n}{\rho_n}))}{(z_n+\rho_n(\zeta-\frac{z_n}{\rho_n}))^m}\zeta^m\to g(\zeta-c)\zeta^m=G(\zeta)$$

We know that $z=0$ is zero of f_n with multiplicity $m+1$, then we get 0 is a zero of $G(\zeta)$ with multiplicity $m+1$ and

$$G^{(m+1)}(0)=\lim_{n\to\infty}G_n^{(m+1)}(0)=m! \tag{3.5}$$

If $G'(\zeta)\equiv\zeta^m$, we derive that $G(\zeta)=\frac{1}{m+1}\zeta^{m+1}$. Hence we obtain $g(\zeta)=\frac{1}{m+1}(\zeta+c)$. It follows that $g^\sharp(0)\le\frac{1}{m+1}<M+1$, a contradiction. Thus $G'(\zeta)\not\equiv\zeta^m$. Using the same argument as in the proof of Case 1, we get

$$G(\zeta)=0\Leftrightarrow G'(\zeta)=\zeta^m \ \mathit{and}\ G'(\zeta)=\zeta^m\Rightarrow\begin{cases}G^{(k)}(\zeta)\le\overline{M}, & \text{k}=m+1,\\ G^{(k)}(\zeta)=0, & \text{k}\ge m+2.\end{cases}$$

Suppose $G(\zeta)$ is a polynomial. Let

$$G(\zeta)=b_q\zeta^q+b_{q-1}\zeta^{q-1}+...+b_{m+1}\zeta^{m+1}\quad(b_{m+1}\neq 0). \tag{3.6}$$

From $G(\zeta)=0\Leftrightarrow G'(\zeta)=\zeta^m$, we get

$$G(\zeta)=\zeta(G'(\zeta)-\zeta^m)A. \tag{3.7}$$

Thus, by (3.6) and (3.7) we have $G(\zeta)=b_q\zeta^q-\frac{1}{q-(m+1)}\zeta^{m+1}$ $(q\ge m+2)$ or $G(\zeta)=A\zeta^{m+1}$, and from (3.5), we get $G(\zeta)=\frac{1}{m+1}\zeta^{m+1}$. Then $G'(\zeta)\equiv\zeta^m$, a contradiction.

In the following we assume that $G(\zeta)$ is a transcendental entire function.

Let us consider the family $T=\{t_n:t_n(\zeta)=\frac{G((2^m)^n\zeta)}{(2^m)^{(m+1)n}}\}$, we see that t_n is a entire function satisfying

$$t_n(\zeta)=0\Leftrightarrow t_n'(\zeta)=\zeta^m\Rightarrow\begin{cases}t_n(\zeta)\le\overline{M}, & \text{k}=m+1,\\ t_n(\zeta)=0, & \text{k}\ge m+2.\end{cases}$$

By Lemma 2.3, we have T is normal on $D_1=\{\zeta:(1/2)^m\le|\zeta|\le 2^m\}$, thus there exists a M_1 satisfying

$$t_n^\sharp(\zeta)=\frac{(2^m)^{(m+2)n}|G'((2^m)^n\zeta)|}{(2^m)^{2(m+1)n}+|(G(2^m)^n\zeta)|^2}\le M_1.$$

Set $r(z)=\frac{G(z)}{z^{m+1}}$, then $r(z)$ is a transcendental entire function. We know that for each $z\in C$, there exists a integer n such that $z=(2^m)^n\zeta$, where $(1/2)^m\le|\zeta|\le 2^m$. We can get

$$|z|r^\sharp(z)\le(2^m)^{3m+4}t_n^\sharp(\zeta)+\frac{m+1}{2}\le(2^m)^{3m+4}M_1+\frac{m+1}{2}. \tag{3.8}$$

From Lemma 2.6, we get

$$\limsup_{|z|\to\infty} |z| r^{\sharp}(z) = +\infty,$$

which contradicts with (3.8).

Thus, we prove that $\mathcal{F}_1$ is normal at $z = 0$. Hence $\mathcal{F}$ is normal at $z = 0$.

4 Proof of Proposition 2

Let $z_0 \in D$. If $[Q(z) - Q'(z)]|_{z=z_0} \neq 0$, by Lemma 2.3, $\mathcal{F}$ is normal at z_0. Now suppose that $[Q(z) - Q'(z)]|_{z=z_0} = 0$. Without loss of generality, we may assume that $z_0 = 0$, $\Delta = \{z : |z| < \delta\} \in D$ and $R(z) = Q(z) - Q'(z) = z^m P(z)$, where $P(z) \neq 0 \ \ (z \in \Delta)$. We shall prove that $\mathcal{F}$ is normal at $z = 0$.

Suppose on the contrary that $\mathcal{F}$ is not normal at $z = 0$, then by Lemma 2.1, we can find $z_n \to 0$, $\rho_n \to 0$ and $f_n \in \mathcal{F}$ such that

$$g_n(\zeta) = f_n(z_n + \rho_n\zeta) \to g(\zeta) \tag{4.1}$$

locally uniformly on C, where g is a nonconstant entire function. Without loss of generality, we assume that

$$\lim_{n\to\infty} \frac{z_n}{\rho_n} = c \in C.$$

First, we shall prove that $g(\zeta)$ is a transcendental entire function. In fact, we only need to prove that $g(\zeta) \neq 0$. The argument given in the proof of Proposition 1 shows that

$$g(\zeta) = 0 \Rightarrow g'(\zeta) = 0,$$

thus g only has multiple zeros. Suppose ζ_0 is a zero of $g(\zeta)$ with multiplicity $s(\geq 2)$, then $g^{(s)}(\zeta_0) \neq 0$. Thus there exists a positive number δ, such that

$$g(\zeta) \neq 0,\ g'(\zeta) \neq 0,\ g^{(s)}(\zeta) \neq 0 \tag{4.2}$$

on $D^o_\delta = \{\zeta : 0 < |\zeta - \zeta_0| < \delta\}$. By (4.1) and Rouché theorem, there exist $\zeta_{n,j} (j = 1, 2, ..., s)$ on $D_{\delta/2} = \{\zeta : |\zeta - \zeta_0| < \delta/2\}$ such that

$$g_n(\zeta_{n,j}) = f_n(z_n + \rho_n\zeta_{n,j}) = 0 \ (j = 1, 2, ..., s).$$

It follows from $R(z) = 0 \Rightarrow f(z) \neq 0$ and $f(z) = 0 \rightleftharpoons f'(z) = R(z)$ that $f_n'(z_n + \rho_n\zeta_{n,j}) = R(z_n + \rho_n\zeta_{n,j}) \neq 0$. Thus

$$g_n'(\zeta_{n,j}) = \rho_n f_n'(z_n + \rho_n\zeta_{n,j}) = \rho_n R(z_n + \rho_n\zeta_{n,j}) \neq 0 (j = 1, 2, ..., s),$$

so each $\zeta_{n,j}$ is a simple zero of $g_n(\zeta)$, that is $\zeta_{n,j} \neq \zeta_{n,i} (1 \leq i \neq j \leq s)$. On the other hand

$$\lim_{n\to\infty} g_n'(\zeta_{n,j}) = \lim_{n\to\infty} \rho_n R(z_n + \rho_n\zeta_{n,j}) = 0$$

From (4.2), we get

$$\lim_{n\to\infty} \zeta_{n,j} = \zeta_0 \ \ (j = 1, 2, ..., s).$$

Noting that (4.2) and $g_n'(\zeta)-\rho_nR(z_n+\rho_n\zeta)$ has s zeros $\zeta_{n,j}(j=1,2,...,s)$ in $D_{\delta/2}$, then ζ_0 is a zero of $g'(\zeta)$ of multiplicity s, and thus $g^{(s)}(\zeta_0)=0$. This is a contradiction. Hence $g(\zeta)\neq 0$ and $g(\zeta)$ is a transcendental entire function.

Now we consider five cases.

Case 1: There exist infinitely many $\{n_j\}$ such that

$$f_{n_j}'(z_{n_j}+\rho_{n_j}\zeta)\equiv R(z_{n_j}+\rho_{n_j}\zeta).$$

It follows that $g_{n_j}'(\zeta)\equiv\rho_{n_j}R(z_{n_j}+\rho_{n_j}\zeta)$. Let $j\to\infty$, we deduce that $g'(\zeta)\equiv 0$, which contradicts that g is transcendental.

Case 2: There exist infinitely many $\{n_j\}$ such that $\psi_{n_j}(z_{n_j}+\rho_{n_j}\zeta)\equiv 0$, where $\psi_n=\frac{Rf_n^{(k)}-Qf_n'}{f_n}$. Thus we have

$$(z_{n_j}+\rho_{n_j}\zeta)^mP(z_{n_j}+\rho_{n_j}\zeta)\frac{g_{n_j}^{(k)}(\zeta)}{\rho_{n_j}^k}\equiv Q(z_{n_j}+\rho_{n_j}\zeta)\frac{g_{n_j}'(\zeta)}{\rho_{n_j}}$$

and

$$\frac{g_{n_j}^{(k)}(\zeta)}{g_{n_j}'(\zeta)}=\frac{Q(z_{n_j}+\rho_{n_j}\zeta)\rho_{n_j}^{(k-(m+1))}}{P(z_{n_j}+\rho_{n_j}\zeta)(\frac{z_{n_j}}{\rho_{n_j}}+\zeta)^m}.$$

Noting that $k\geq 2q+1\geq 2m+1$, let $j\to\infty$, we deduce that $g^{(k)}(\zeta)\equiv 0$, which contradicts that g is transcendental.

Case 3: There exist infinitely many $\{n_j\}$ such that $\varphi_{n_j}(z_{n_j}+\rho_{n_j}\zeta)\equiv 0$, where

$$\varphi_n=-[1+(\frac{1}{\psi_n})'Q+\frac{1}{\psi_n}Q']R-\frac{1}{\psi_n}QR'.$$

and ψ_n is defined as above. Let

$$\begin{aligned}\Gamma(\zeta)=\rho_{n_j}^{k-(m+1)}[(\frac{z_{n_j}}{\rho_{n_j}}+\zeta)^{m-1}P_1(z_{n_j}+\rho_{n_j}\zeta)g_{n_j}^{(k)}(\zeta)+(\frac{z_{n_j}}{\rho_{n_j}}+\zeta)^mP(z_{n_j}+\rho_{n_j}\zeta)g_{n_j}^{(k+1)}(\zeta)\\ -\rho_{n_j}^{k-m}Q'(z_{n_j}+\rho_{n_j}\zeta)g_{n_j}'(\zeta)-\rho_{n_j}^{k-(m+1)}Q(z_{n_j}+\rho_{n_j}\zeta)g_{n_j}''(\zeta)].\end{aligned}$$

Then

$$\begin{aligned}&\frac{-\Gamma(\zeta)Q(z_{n_j}+\rho_{n_j}\zeta)}{(\frac{z_{n_j}}{\rho_{n_j}}+\zeta)^mP(z_{n_j}+\rho_{n_j}\zeta)g_{n_j}^{(k)}(\zeta)-Q(z_{n_j}+\rho_{n_j}\zeta)\rho_{n_j}^{k-(m+1)}g_{n_j}'(\zeta)}+Q'(z_{n_j}+\rho_{n_j}\zeta)\rho_{n_j}^{k-m}\\ &+(\frac{z_{n_j}}{\rho_{n_j}}+\zeta)^mP(z_{n_j}+\rho_{n_j}\zeta)\frac{g_{n_j}^{(k)}(\zeta)}{g_{n_j}(\zeta)}=\frac{Q(z_{n_j}+\rho_{n_j}\zeta)\rho_{n_j}^{k-(m+1)}P_1(z_{n_j}+\rho_{n_j}\zeta)}{(\frac{z_{n_j}}{\rho_{n_j}}+\zeta)P(z_{n_j}+\rho_{n_j}\zeta)},\end{aligned}$$

where $R'(z)=z^{m-1}P_1(z)$.

Thus, let $j\to\infty$, we get $g^{(k)}(\zeta)\equiv 0$, which contradicts that g is transcendental.

Case 4: There exist infinitely many $\{n_j\}$ such that $\mu_{kn_j}(z_{n_j}+\rho_{n_j}\zeta)\equiv Q(z_{n_j}+\rho_{n_j}\zeta)$ where

$$\mu_{kn}=R\{\lambda_n^{k-1}+P_{k-2}[\lambda_n]\}+R'\{\lambda_n^{k-2}+P_{k-3}[\lambda_n]\}+...+R^{(q)}\{\lambda_n^{k-(q+1)}+P_{k-(q+2)}[\lambda_n]\},$$

and $\lambda_n=\frac{f_n'-R}{f_n}$. Thus, let $j\to\infty$, we get

$$(\frac{g'}{g})^{k-(m+1)}[(c+\zeta)^mP(0)(\frac{g'}{g})^m+R^{(m)}(0)]\equiv 0.$$

Hence $g' \equiv 0$ or $(c+\zeta)^m P(0)(\frac{g'}{g})^m + R^{(m)}(0) \equiv 0$, which contradicts that g is a transcendental entire function.

Case 5: There exist finitely many $\{n_j\}$ such that $f'_{n_j}(z_{n_j}+\rho_{n_j}\zeta) \equiv R(z_{n_j}+\rho_{n_j}\zeta)$, $\psi_{n_j}(z_{n_j}+\rho_{n_j}\zeta) \equiv 0$, $\varphi_{n_j}(z_{n_j}+\rho_{n_j}\zeta) \equiv 0$ and $\mu_{kn_j}(z_{n_j}+\rho_{n_j}\zeta) \equiv Q(z_{n_j}+\rho_{n_j}\zeta)$.

For all n we may suppose that $f'_{n_j}(z_{n_j}+\rho_{n_j}\zeta) \not\equiv R(z_{n_j}+\rho_{n_j}\zeta)$, $\psi_{n_j}(z_{n_j}+\rho_{n_j}\zeta) \not\equiv 0$, $\varphi_{n_j}(z_{n_j}+\rho_{n_j}\zeta) \not\equiv 0$ and $\mu_{kn_j}(z_{n_j}+\rho_{n_j}\zeta) \not\equiv Q(z_{n_j}+\rho_{n_j}\zeta)$.

Take $\zeta_0 \in C$ such that $g^{(j)}(\zeta_0) \neq 0$ $(j = 0, 1, ..., k)$. In case $c \neq \infty$, choose ζ_0 to satisfy the additional conditions that $\zeta_0 \neq -c$ and

$$(c+\zeta_0)^m P(0)(\frac{g'(\zeta_0)}{g(\zeta_0)})^m + R^{(m)}(0) \neq 0.$$

Noting that $k \geq 2q+1 \geq 2m+1$, this facts imply that $K_n \Longrightarrow 0$ as $n \to \infty$, so that $\log K_n \to -\infty$ as $n \to \infty$.

For $n = 1, 2, 3, ...,$ put

$$h_n(z) = f_n(z_n + \rho_n\zeta_0 + z)$$

Since $z_n + \rho_n\zeta_0 \to 0$ as $n \to \infty$, it follows that (for sufficiently large n) h_n is defined and holomorphic on $|z| < \frac{1}{2}$. Denote

$$a_n = z_n + \rho_n\zeta_0.$$

Then, for sufficiently large n, $h_n(0) \neq 0$, $h'_n(0) - R_{a_n}(0) \neq 0$. By the assumption we get

$$h_n(z) = 0 \rightleftharpoons h'_n(z) = R_{a_n} \Rightarrow h_n^{(k)}(z) = Q_{a_n}.$$

Let $a = a_n$ and $f(z) = h_n(z)$ in Lemma 2.4, then we get

$$h_n(-a_n) = f_n(0) \neq 0, \psi_{a_n}(0) = \psi_n(a_n) \neq 0,\ \varphi_{a_n}(0) = \varphi_n(a_n) \neq 0,$$

$$[\mu_{a_n k} - Q_{a_n}]|_{z=0} = [\mu_{kn} - Q]|_{z=a_n} \neq 0,$$

thus $h_n(z)$ satisfies the assumption of Lemma 2.4.

Now applying Lemma 2.4 with $r_0 = \frac{1}{2}$, and noting that the last three terms in (2.4) are bounded for $0 < r < 1/3$, we obtain that, for sufficiently large n and $0 < r < 1/3$,

$$\begin{aligned} T(r,h_n) \leq M_1[m(r,\frac{h'_n}{h_n}) + m(r,\frac{h_n^{(k)}}{h_n}) + m(r,\frac{h_n^{(k)}}{h'_n}) + m(r,\frac{h''_n - R'_{a_n}}{h'_n - R_{a_n}}) + m(r,\frac{h_n^{(k)}}{h'_n - R_{a_n}})] \\ + M_1 m(r,\frac{h_n^{(k+1)}}{h'_n - R_{a_n}}) + M_2[m(r,\frac{\psi'_{a_n}}{\psi_{a_n}}) + m(r,\frac{\lambda'_{a_n}}{\lambda_{a_n}}) + ... + m(r,\frac{\lambda_{a_n}^{(k-2)}}{\lambda_{a_n}})]. \end{aligned}$$

We can obtain, for $0 < r < \tau < 1/3$,

$$\begin{aligned} T(r,h_n) \leq C_k\{1 + \log^+\frac{1}{r} + \log^+\frac{1}{\tau - r} + \log^+ T(\tau,h_n) \\ + \log^+ T(\tau,h'_n) + \log^+ T(\tau,\psi_{a_n}) + \log^+ T(\tau,\lambda_{a_n})\}. \end{aligned} \tag{4.3}$$

Observe that $T(\tau,h'_n) = m(\tau,h'_n) \leq m(\tau,h_n) + m(r,\frac{h'_n}{h_n})$, hence for $1/4 < r < \rho < 1/3$ with $\tau = (r+\rho)/2$. From the above we obtain

$$T(r,h_n) \leq C_k(1 + \log^+\frac{1}{\rho - r} + \log^+ T(\rho,h_n)).$$

By Lemma 2.5 it then follows that $T(1/4, h_n) \leq A$, where A is a constant independent of n. Thus $f_n(z)$ is uniformly bounded for sufficiently large n and $|z| < 1/8$. However, from $\rho_n^2 f_n''(z_n + \rho_n\zeta_0) = g_n''(\zeta_0) \to g''(\zeta_0) \neq 0$ we see that $f(z)$ cannot bounded in $|z| < 1/8$. This is a contradiction, so the proof is complete.

5 Proof of Theorem 1.1

Let $\mathcal{G} = \{g = f - Q : f \in \mathcal{F}\}$ and $R(z) = Q(z) - Q'(z)$. Obviously, $\mathcal{G}$ is normal in D if and only if $\mathcal{F}$ is normal in D. It follows from our assumption that, for any $g \in \mathcal{G}$, we have

$$g = 0 \rightleftharpoons g' = R \Rightarrow g^{(k)} = Q. \tag{5.1}$$

Let $z_0 \in D$. Now we prove that $\mathcal{G}$ is normal at z_0. Let $\{g_n\} \subset \mathcal{G}$ be a sequence.

If $R(z_0) \neq 0$, then there exists a positive number δ such that $\Delta_\delta = \{z \in D : |z - z_0| < \delta\} \subset D$ and $R(z) \neq 0$ in Δ_δ. Then by Lemma 2.3, $\{g_n\}$ is normal at z_0.

If $R(z_0) = 0$, then there exists a positive number δ such that $\Delta_\delta = \{z \in D : |z - z_0| < \delta\} \subset D$ and $R(z) \neq 0$ in $\Delta_\delta \backslash \{z_0\}$. Suppose $\{g_n\}$ has a subsequence say, without loss of generality, itself, such that $g_n(z_0) = 0$, then $\{g_n\}$ is normal at z_0 by Proposition 1. Suppose $g_n(z_0) \neq 0$ for all but finite many of $\{g_n\}$, then $\{g_n\}$ is normal at z_0 by Proposition 2.

Thus $\mathcal{F}$ is normal in D and hence Theorem 1.1 is proved.

Bibliography

1. F. Bureau. Mémoire sur les fonctions uniformes à point singuliar essentiel isolé, Mém. Soc. Roy. Sci. Liége(3). 17, 1932.

2. J.M. Chang, M.L. Fang and L. Zalcman. Normal families of holomorphic functions, Illinois Journal of Mathematics. 48(2004), 319-337.

3. J. M. Chang, M. L. Fang. Normality and shared functions of holomorphic functions and their derivatives, Michigan Math. J. 53(2005), 625-645.

4. J. Clunie, W.K. Hayman. The spherical derivative of integral and meromorphic functions, Comment Math. Helvet. 40(1966), 117-148.

5. K.L. Hiong. Sur les fonctions holomorphes dont les dérivées admettent une valeur exceptionnelle, Ann. Sci. École. 72(1955), 165-197.

6. J. Grahl, C. Meng. Entire functions sharing a polynomial with their derivatives and normal families, Analysis., 28 (2008), 51–61.

7. X.C. Pang, L. Zalcman. Normal families and shared values, Bull.London Math.Soc. 32(2000), 325-331.

8. W. Schwick. Sharing values and normality, Arch. Math.(Basel) 59 (1992), 50-54.

9. J.P. Wang, H.X. Yi. Entire functions that share a polynomial with their derivatives. J.Math.Anal.Appl. 320 (2006), 703-717.

10. H.X. Yi, C.C. Yang. Uniqueness theory of meromorphic functions, Science Press, Beijing, 1995.

НОРМАЛЬНОЕ СЕМЕЙСТВО И РАСПРЕДЕЛЕННЫЕ МНОГОЧЛЕНЫ МЕРОМОРФНЫХ ФУНКЦИЙ

Фенг Лю[1], Жан Фенг Ксю[2]

[1] Китайский нефтяной университет,
Донгиин, Шаньдун 257061, Китай, e-mail: xujunf@gmail.com

[2] Вууй Университет,
Цзянмэнь, Гуангдонг, 529020, Китай, e-mail: lvfeng@mail.sdu.edu.cn

Аннотация. В работе изучается единственность и разделение неподвижной точки мероморфных функций. Доказаны две основные теоремы, улучшающие результаты Ванга и Кью.

Ключевые слова: мероморфная функция, неподвижная точка, распределенные многочлены.

УДК 517.55

SPARSE HYPERGEOMETRIC SYSTEMS

Timur Sadykov

Institute of Mathematics, Siberian Federal University,
pr. Svobodny, 79, Krasnoyarsk, 660041, Russia, e-mail: sadykov@lan.krasu.ru

Abstract. We study the approach to the theory of hypergeometric functions in several variables via a generalization of the Horn system of differential equations. A formula for the dimension of its solution space is given. Using this formula we construct an explicit basis in the space of holomorphic solutions to the generalized Horn system under some assumptions on its parameters.

Keywords: hypergeometric functions, Horn system of differential equations, Mellin system.

1 Introduction

There exist several approaches to the notion of a hypergeometric function depending on several complex variables. It can be defined as the sum of a power series of a certain form (such series are known as Γ-series) [10], as a solution to a system of partial differential equations [9], [11], [1], or as a Mellin-Barnes integral [15]. In the present paper we study the approach to the theory of hypergeometric functions via a generalization of the Horn system of differential equations. We consider the system of partial differential equations of hypergeometric type

$$x^{u_i} P_i(\theta) y(x) = Q_i(\theta) y(x), \ i = 1, \dots, n, \tag{1.1}$$

where the vectors $u_i = (u_{i1}, \dots, u_{in}) \in \mathbb{Z}^n$ are assumed to be linearly independent, P_i, Q_i are nonzero polynomials in n complex variables and $\theta = (\theta_1, \dots, \theta_n)$, $\theta_i = x_i \frac{\partial}{\partial x_i}$. We use the notation $x^{u_i} = x_1^{u_{i1}} \dots x_n^{u_{in}}$. If $\{u_i\}_{i=1}^n$ form the standard basis of the lattice $\mathbb{Z}^n$ then the system (1.1) coincides with a classical system of partial differential equations which goes back to Horn and Mellin (see [13] and § 1.2 of [10]). In the present paper the system (1.1) is referred to as the *sparse hypergeometric system* (or generalized Horn system) since, in general, its series solutions might have many gaps.

A sparse hypergeometric system can be easily reduced to the classical Horn system by a monomial change of variables. The main purpose of the present paper is to discuss the relation between the sparse and the classical case in detail for the benefit of a reader interested in explicit solutions of hypergeometric $\mathcal{D}$-modules. We also furnish several examples which illustrate crucial properties of the singularities of multivariate hypergeometric functions. Most of the statements in this article are parallel to or follow from the results in [16].

A typical example of a sparse hypergeometric system is the Mellin system of equations (see [7]). One of the reasons for studying sparse hypergeometric systems is the fact that knowing the structure of solutions to (1.1) allows one to investigate the so-called amoeba of the singular locus of a solution to (1.1). The notion of amoebas was introduced by Gelfand, Kapranov and Zelevinsky (see [12], Chapter 6, § 1). Given a mapping $f(x)$, its amoeba $\mathcal{A}_f$ is the image of the hypersurface $f^{-1}(0)$ under the map $(x_1, \dots, x_n) \mapsto (\log|x_1|, \dots, \log|x_n|)$. In section 5 we use the

The author was supported by the Russian Foundation for Basic Research, grant 09-01-00762-a, by grant no. 26 for scientific research groups of Siberian Federal University and by the "Dynasty"foundation.

results on the structure of solutions to (1.1) for computing the number of connected components of the complement of amoebas of some rational functions. The problem of describing the class of rational hypergeometric functions was studied in a different setting in [5], [6]. The definition of a hypergeometric function used in these papers is based on the Gelfand-Kapranov-Zelevinsky system of differential equations [9], [10], [11].

Solutions to (1.1) are closely related to the notion of a generalized Horn series which is defined as a formal (Laurent) series

$$y(x) = x^{\gamma} \sum_{s \in \mathbb{Z}^n} \varphi(s) x^s, \tag{1.2}$$

whose coefficients $\varphi(s)$ are characterized by the property that $\varphi(s+u_i) = \varphi(s) R_i(s)$. Here $R_i(s)$ are rational functions. We also use notations $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{C}^n$, $\operatorname{Re}\gamma_i \in [0,1)$, $x^s = x_1^{s_1} \ldots x_n^{s_n}$. In the case when $\{u_i\}_{i=1}^n$ form the standard basis of $\mathbb{Z}^n$ we get the definition of the classical Horn series (see [10], § 1.2).

In the case of two or more variables the generalized Horn system (1.1) is in general not solvable in the class of series (1.2) without additional assumptions on the polynomials P_i, Q_i. In section 2 we investigate solvability of hypergeometric systems of equations and describe supports of solutions to the generalized Horn system. The necessary and sufficient conditions for a formal solution to the system (1.1) in the class (1.2) to exist are given in Theorem 2.1.

In section 3 we consider the $\mathcal{D}$-module associated with the generalized Horn system. We give a formula which allows one to compute the dimension of the space of holomorphic solutions to (1.1) at a generic point under some additional assumptions on the system under study (Theorem 3.3). We give also an estimate for the dimension of the solution space of (1.1) under less restrictive assumptions on the parameters of the system (Corollary 3.4).

In section 4 we consider the case when the polynomials P_i, Q_i can be factorized up to polynomials of degree 1 and construct an explicit basis in the space of holomorphic solutions to some systems of the Horn type. We show that in the case when $R_i(s+u_j)R_j(s) = R_j(s+u_i)R_i(s)$, $Q_i(s+u_j) = Q_i(s)$ and $\deg Q_i(s) > \deg P_i(s)$, $i,j = 1, \ldots, n$, $i \neq j$, there exists a basis in the space of holomorphic solutions to (1.1) consisting of series (1.2) if the parameters of the system under study are sufficiently general (Theorem 4.1).

In section 5 we apply the results on the generalized Horn system to the problem of describing the complement of the amoeba of a rational function. We show how Theorem 2.1 can be used for studying Laurent series developments of a rational solution to (1.1). A class of rational hypergeometric functions with minimal number of connected components of the complement of the amoeba is described.

2 Supports of solutions to sparse hypergeometric systems

Suppose that the series (1.2) represents a solution to the system (1.1). Computing the action of the operator $x^{u_i} P_i(\theta) - Q_i(\theta)$ on this series we arrive at the following system of difference equations

$$\varphi(s+u_i) Q_i(s+\gamma+u_i) = \varphi(s) P_i(s+\gamma), \ i = 1, \ldots, n. \tag{2.1}$$

The system (2.1) is equivalent to (1.1) as long as we are concerned with those solutions to the generalized Horn system which admit a series expansion of the form (1.2). Let $\mathbb{Z}^n + \gamma$ denote the shift in $\mathbb{C}^n$ of the lattice $\mathbb{Z}^n$ with respect to the vector γ. Without loss of generality we assume that the polynomials $P_i(s), Q_i(s+u_i)$ are relatively prime for all $i = 1, \ldots, n$. In this section we shall describe nontrivial solutions to the system (2.1) (i.e. those ones which are not equal to zero identically). While looking for a solution to (2.1) which is different from zero on some subset S of $\mathbb{Z}^n$ we shall assume that the polynomials $P_i(s), Q_i(s)$, the set S and the vector γ satisfy the condition

$$|P_i(s+\gamma)| + |Q_i(s+\gamma+u_i)| \neq 0, \tag{2.2}$$

for any $s \in S$ and for all $i = 1, \ldots, n$. That is, for any $s \in S$ the equality $P_i(s+\gamma) = 0$ implies that $Q_i(s+\gamma+u_i) \neq 0$ and $Q_i(s+\gamma+u_i) = 0$ implies $P_i(s+\gamma) \neq 0$.

The system of difference equations (2.1) is in general not solvable without further restrictions on P_i, Q_i. Let $R_i(s)$ denote the rational function $P_i(s)/Q_i(s+u_i)$, $i = 1, \ldots, n$. Increasing the argument s in the ith equation of (2.1) by u_j and multiplying the obtained equality by the jth equation of (2.1), we arrive at the relation $\varphi(s+u_i+u_j)/\varphi(s) = R_i(s+u_j)R_j(s)$. Analogously, increasing the argument in the jth equation of (2.1) by u_i and multiplying the result by the ith equation of (2.1), we arrive at the equality $\varphi(s+u_i+u_j)/\varphi(s) = R_j(s+u_i)R_i(s)$. Thus the conditions

$$R_i(s+u_j)R_j(s) = R_j(s+u_i)R_i(s), \;\; i, j = 1, \ldots, n \tag{2.3}$$

are in general necessary for (2.1) to be solvable. The conditions (2.3) will be referred to as the compatibility conditions for the system (2.1). Throughout this paper we assume that the polynomials P_i, Q_i defining the generalized Horn system (1.1) satisfy (2.3).

Let U denote the matrix whose rows are the vectors $u_1, \ldots, u_n$. A set $S \subset \mathbb{Z}^n$ is said to be *U-connected* if any two points in S can be connected by a polygonal line with the vectors $u_1, \ldots, u_n$ as sides and vertices in S. Let $\varphi(s)$ be a solution to (2.1). We define the *support* of $\varphi(s)$ to be the subset of the lattice $\mathbb{Z}^n$ where $\varphi(s)$ is different from zero. A formal series $x^\gamma \sum_{s \in \mathbb{Z}^n} \varphi(s) x^s$ is called a *formal solution* to the system (1.1) if the function $\varphi(s)$ satisfies the equations (2.1) at each point of the lattice $\mathbb{Z}^n$. The following Theorem gives necessary and sufficient conditions for a solution to the system (2.1) supported in some set $S \subset \mathbb{Z}^n$ to exist.

Theorem 2.1 *For $S \subset \mathbb{Z}^n$ define*

$$S_i^{'} = \{s \in S : s + u_i \notin S\}, \;\; S_i^{''} = \{s \notin S : s + u_i \in S\}, \;\; i = 1, \ldots, n.$$

Suppose that the conditions (2.2) are satisfied on S. Then there exists a solution to the system (2.1) supported in S if and only if the following conditions are fulfilled:

$$P_i(s+\gamma)|_{S_i^{'}} = 0, \;\; Q_i(s+\gamma+u_i)|_{S_i^{''}} = 0, \;\; i = 1, \ldots, n, \tag{2.4}$$

$$P_i(s+\gamma)|_{S \setminus S_i^{'}} \neq 0, \;\; Q_i(s+\gamma+u_i)|_S \neq 0, \;\; i = 1, \ldots, n. \tag{2.5}$$

The proof of this theorem is analogous to the proof of Theorem 1.3 in [16]. Theorem 2.1 will be used in section 4 for constructing an explicit basis in the space of holomorphic solutions to the generalized Horn system in the case when $\deg Q_i > \deg P_i$ and $Q_i(s+u_j) = Q_i(s)$, $i, j = 1, \ldots, n$, $i \neq j$. In the next section we compute the dimension of the space of holomorphic solutions to (1.1) at a generic point.

3 Holomorphic solutions to sparse systems

Let G_i denote the differential operator $x^{u_i}P_i(\theta)-Q_i(\theta)$, $i=1,\ldots,n$. Let $\mathcal{D}$ be the Weyl algebra in n variables [3], and define $\mathcal{M}=\mathcal{D}/\sum_{i=1}^n\mathcal{D}G_i$ to be the left $\mathcal{D}$-module associated with the system (1.1). Let $R=\mathbb{C}[z_1,\ldots,z_n]$ and $R[x]=R[x_1,\ldots,x_n]=\mathbb{C}[x_1,\ldots,x_n,z_1,\ldots,z_n]$. We make $R[x]$ into a left $\mathcal{D}$-module by defining the action of ∂_j on $R[x]$ by

$$\partial_j=\frac{\partial}{\partial x_j}+z_j. \tag{3.1}$$

Let $\Phi:\mathcal{D}\to R[x]$ be the $\mathcal{D}$-linear map defined by

$$\Phi(x_1^{a_1}\ldots x_n^{a_n}\partial_1^{b_1}\ldots\partial_n^{b_n})=x_1^{a_1}\ldots x_n^{a_n}z_1^{b_1}\ldots z_n^{b_n}. \tag{3.2}$$

It is easily checked that Φ is an isomorphism of $\mathcal{D}$-modules. In this section we establish some properties of linear operators acting on $R[x]$. We aim to construct a commutative family of $\mathcal{D}$-linear operators $W_i:R[x]\to R[x]$, $i=1,\ldots,n$ which satisfy the equality $\Phi(G_i)=W_i(1)$. The crucial point which requires additional assumptions on the parameters of the system (1.1) is the commutativity of the family $\{W_i\}_{i=1}^n$ which is needed for computing the dimension (as a $\mathbb{C}$-vector space) of the module $R[x]/\sum_{i=1}^n W_iR[x]$ at a fixed point $x^{(0)}$. We construct the operators W_i and show that they commute with one another under some additional assumptions on the polynomials $Q_i(s)$ (Lemma 3.1). However, no additional assumptions on the polynomials $P_i(s)$ are needed as long as the compatibility conditions (2.3) are fulfilled.

Following the spirit of Adolphson [1] we define operators $D_i:R[x]\to R[x]$ by setting

$$D_i=z_i\frac{\partial}{\partial z_i}+x_iz_i,\ i=1,\ldots,n. \tag{3.3}$$

It was pointed out in [1] that the operators (3.3) form a commutative family of $\mathcal{D}$-linear operators. Let D denote the vector $(D_1,\ldots,D_n)$. For any $i=1,\ldots,n$ we define operator $\nabla_i:R[x]\to R[x]$ by $\nabla_i=z_i^{-1}D_i$. This operator commutes with the operators ∂_j since both D_i and the multiplication by z_i^{-1} commute with ∂_j. Moreover, the operator ∇_i commutes with ∇_j for all $1\le i,j\le n$ and with D_j for $i\ne j$. In the case $i=j$ we have $\nabla_iD_i=\nabla_i+D_i\nabla_i$.

Thanks to Lemma 2.2 in [16] we may define operators $W_i=P_i(D)\nabla^{u_i}-Q_i(D)$ such that for any $i=1,\ldots,n$ W_i is a $\mathcal{D}$-linear operator satisfying the identity $\Phi(G_i)=W_i(1)$. It follows by the $\mathcal{D}$-linearity of W_i that $\sum_{i=1}^n W_iR[x]$ and $R[x]/\sum_{i=1}^n W_iR[x]$ can be considered as left $\mathcal{D}$-modules. Using Theorem 4.4 and Lemma 4.12 in [1], we conclude that the following isomorphism holds true:

$$\mathcal{M}\simeq R[x]\Bigg/\left(\sum_{j=1}^n W_jR[x]\right). \tag{3.4}$$

In the general case the operators $W_i=P_i(D)\nabla^{u_i}-Q_i(D)$ do not commute since D_i does not commute with ∇_i. However, this family of operators may be shown to be commutative under some assumptions on the polynomials $Q_i(s)$ in the case when the polynomials $P_i(s),Q_i(s)$ satisfy the compatibility conditions (2.3). The following Lemma holds.

Lemma 3.1 *The operators $W_i=P_i(D)\nabla^{u_i}-Q_i(D)$ commute with one another if and only if the polynomials $P_i(s),Q_i(s)$ satisfy the compatibility conditions (2.3) and for any $i,j=1,\ldots,n$, $i\ne j$, $Q_i(s+u_j)=Q_i(s)$.*

Proof Since $\nabla_i = z_i^{-1} + D_i z_i^{-1}$ it follows that $\nabla_i D_i = \nabla_i + D_i \nabla_i$ and that ∇_i commutes with D_j for $i \neq j$. Hence for any $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$

$$\nabla_i D_1^{\alpha_1} \ldots D_n^{\alpha_n} = D_1^{\alpha_1} \ldots (D_i + 1)^{\alpha_i} \ldots D_n^{\alpha_n} \nabla_i. \tag{3.5}$$

Let E_i^t denote the operator which increases the ith argument by t, that is, $E_i^t f(x) = f(x + te_i)$. Here $\{e_i\}_{i=1}^n$ denotes the standard basis of $\mathbb{Z}^n$. It follows from (3.5) that

$$\nabla_i P_j(D) = (E_i^1 P_j)(D) \nabla_i. \tag{3.6}$$

For $\alpha \in \mathbb{Z}^n$ let E^α denote the composition $E_1^{\alpha_1} \circ \ldots \circ E_n^{\alpha_n}$. Using (3.6) we compute the commutator of the operators W_i, W_j :

$$W_i W_j - W_j W_i = \Big(P_i(D)(E^{u_i} P_j)(D) - P_j(D)(E^{u_j} P_i)(D) \Big) \nabla^{u_i + u_j} +$$

$$\Big((E^{u_j} Q_i)(D) - Q_i(D) \Big) P_j(D) \nabla^{u_j} + \Big(Q_j(D) - (E^{u_i} Q_j)(D) \Big) P_i(D) \nabla^{u_i}. \tag{3.7}$$

Let us define the grade $g(x^\alpha z^\beta)$ of an element $x^\alpha z^\beta$ of the ring $R[x]$ to be $\alpha - \beta$. Notice that $g(D_i(x^\alpha z^\beta)) = \alpha - \beta$ and that $g(\nabla_i(x^\alpha z^\beta)) = \alpha - \beta + e_i$, for any $\alpha, \beta \in \mathbb{N}_0^n$. The result of the action of the operator in the right-hand side of (3.7) on $x^\alpha z^\beta$ consists of three terms whose grades are $\alpha - \beta + u_i + u_j,\ \alpha - \beta + u_j$ and $\alpha - \beta + u_i$. Thus the operators W_i, W_j commute if and only if

$$Q_i(D) = (E^{u_j} Q_i)(D),\ i, j = 1, \ldots, n, \quad i \neq j, \tag{3.8}$$

and

$$P_i(D)(E^{u_i} P_j)(D) = P_j(D)(E^{u_j} P_i)(D),\ i, j = 1, \ldots, n. \tag{3.9}$$

It follows from (3.8) that the condition $Q_i(s + u_j) = Q_i(s)$, $i, j = 1, \ldots, n$, $i \neq j$ is necessary for the family $\{W_i\}_{i=1}^n$ to be commutative. Under this assumption on the polynomials $Q_i(s)$ the compatibility conditions (2.3) can be written in the form

$$P_i(s + u_j) P_j(s) = P_j(s + u_i) P_i(s),\ i, j = 1, \ldots, n$$

and they are therefore equivalent to (3.9). The proof is complete.

For $x^{(0)} \in \mathbb{C}^n$ let $\hat{\mathcal{O}}_{x^{(0)}}$ be the $\mathcal{D}$-module of formal power series centered at $x^{(0)}$. Let $\mathbb{C}_{x^{(0)}}$ denote the set of complex numbers $\mathbb{C}$ considered as a $\mathbb{C}[x_1, \ldots, x_n]$-module via the isomorphism $\mathbb{C} \simeq \mathbb{C}[x_1, \ldots, x_n]/(x_1 - x_1^{(0)}, \ldots, x_n - x_n^{(0)})$. We use the following isomorphism (see Proposition 2.5.26 in [4] or [1], § 4) between the space of formal solutions to $\mathcal{M}$ at $x^{(0)}$ and the dual space of $\mathbb{C}_{x^{(0)}} \otimes_{\mathbb{C}[x]} \mathcal{M}$

$$\mathrm{Hom}_{\mathcal{D}}(\mathcal{M}, \hat{\mathcal{O}}_{x^{(0)}}) \simeq \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}_{x^{(0)}} \otimes_{\mathbb{C}[x]} \mathcal{M}, \mathbb{C}). \tag{3.10}$$

This isomorphism holds for any finitely generated $\mathcal{D}$-module. Using (3.4) and fixing the point $x = x^{(0)}$ we arrive at the isomorphism

$$\mathbb{C}_{x^{(0)}} \otimes_{\mathbb{C}[x]} \left(R[x] \Big/ \sum_{i=1}^n W_i R[x] \right) \simeq R \Big/ \sum_{i=1}^n W_{i, x^{(0)}} R, \tag{3.11}$$

where $W_{i,x^{(0)}}$ are obtained from the operators W_i by setting $x = x^{(0)}$. Combining (3.10) with (3.11) we see that

$$\mathrm{Hom}_{\mathcal{D}}(\mathcal{M}, \hat{\mathcal{O}}_{x^{(0)}}) \simeq \mathrm{Hom}_{\mathbb{C}}\left(R \Big/ \sum_{i=1}^{n} W_{i,x^{(0)}} R, \ \mathbb{C} \right).$$

Thus the following Lemma holds true.

Lemma 3.2 *The number of linearly independent formal power series solutions to the system (1.1) at the point $x = x^{(0)}$ is equal to* $\dim_{\mathbb{C}} R / \sum_{i=1}^{n} W_{i,x^{(0)}} R$.

For any differential operator $P \in \mathcal{D}$, $P = \sum_{|\alpha| \leq m} c_\alpha(x) \left(\frac{\partial}{\partial x}\right)^\alpha$ its principal symbol $\sigma(P)(x,z) \in R[x]$ is defined by $\sigma(P)(x,z) = \sum_{|\alpha| = m} c_\alpha(x) z^\alpha$. Let $H_i(x,z) = \sigma(G_i)(x,z)$ be the principal symbols of the differential operators which define the generalized Horn system (1.1). Let $J \subset \mathcal{D}$ be the left ideal generated by $G_1, \ldots, G_n$. By the definition (see [3], Chapter 5, § 2) the characteristic variety $\mathrm{char}(\mathcal{M})$ of the generalized Horn system is given by

$$\mathrm{char}(\mathcal{M}) = \{(x,z) \in \mathbb{C}^{2n} : \sigma(P)(x,z) = 0, \ \text{ for all } P \in J\}.$$

Let us define the set $U_{\mathcal{M}} \subset \mathbb{C}^n$ by $U_{\mathcal{M}} = \{x \in \mathbb{C}^n : \exists\, z \neq 0 \text{ such that } (x,z) \in \mathrm{Char}(\mathcal{M})\}$. Theorem 7.1 in [3, Chapter 5] yields that for $x^{(0)} \notin U_{\mathcal{M}}$

$$\mathrm{Hom}_{\mathcal{D}}(\mathcal{M}, \hat{\mathcal{O}}_{x^{(0)}}) \simeq \mathrm{Hom}_{\mathcal{D}}(\mathcal{M}, \mathcal{O}_{x^{(0)}}).$$

It follows from [18] (pages 146,148) that the $\mathbb{C}$-dimension of the factor of the ring R with respect to the ideal generated by the regular sequence of homogeneous polynomials $H_1(x^{(0)}, z), \ldots, H_n(x^{(0)}, z)$ is equal to the product $\prod_{i=1}^{n} \deg H_i(x^{(0)}, z)$. Since a sequence of n homogeneous polynomials in n variables is regular if and only if their common zero is the origin, it follows that $U_{\mathcal{M}} = \emptyset$ in our setting. Using Lemmas 3.1,3.2, and Lemma 2.7 in [16], we arrive at the following Theorem.

Theorem 3.3 *Suppose that the polynomials $P_i(s), Q_i(s)$ satisfy the compatibility conditions (2.3) and that $Q_i(s + u_j) = Q_i(s)$ for any $i, j = 1, \ldots, n$, $i \neq j$. If the principal symbols $H_1(x^{(0)}, z), \ldots, H_n(x^{(0)}, z)$ of the differential operators $G_1, \ldots, G_n$ form a regular sequence at $x^{(0)}$ then the dimension of the space of holomorphic solutions to (1.1) at the point $x^{(0)}$ is equal to* $\prod_{i=1}^{n} \deg H_i(x^{(0)}, z)$.

Using Lemma 2.7 in [16], we obtain the following result.

Corollary 3.4 *Suppose that the principal symbols $H_1(x^{(0)}, z), \ldots, H_n(x^{(0)}, z)$ of the differential operators $G_1, \ldots, G_n$ form a regular sequence at $x^{(0)}$. Then the dimension of the space of holomorphic solutions to (1.1) at the point $x^{(0)}$ is less than or equal to* $\prod_{i=1}^{n} \deg H_i(x^{(0)}, z)$.

In the next section we, using Theorem 3.3, construct an explicit basis in the space of holomorphic solutions to the generalized Horn system under the assumption that P_i, Q_i can be represented as products of linear factors and that $\deg Q_i > \deg P_i$, $i = 1, \ldots, n$.

4 Explicit basis in the solution space of a sparse hypergeometric system

Throughout this section we assume that the polynomials $P_i(s), Q_i(s)$ defining the generalized Horn system (1.1) can be factorized up to polynomials of degree one. Suppose that $P_i(s), Q_i(s)$ satisfy the following conditions: $Q_i(s+u_j) = Q_i(s)$ and $\deg Q_i > \deg P_i$ for any $i, j = 1, \ldots, n$, $i \neq j$. In this section we will show how to construct an explicit basis in the solution space of such a system of partial differential equations under some additional assumptions which are always satisfied if the parameters of the system under study are sufficiently general.

Recall that U denotes the matrix whose rows are $u_1, \ldots, u_n$ and let U^T denote the transpose of U. Let $\Lambda = (U^T)^{-1}$, let $(\Lambda s)_i$ denote the ith component of the vector Λs and $d_i = \deg Q_i$. Under the above conditions the polynomials $Q_i(s)$ can be represented in the form

$$Q_i(s) = \prod_{j=1}^{d_i} ((\Lambda s)_i - \alpha_{ij}), \quad i = 1, \ldots, n, \quad \alpha_{ij} \in \mathbb{C}.$$

By the Ore–Sato theorem [17] (see also § 1.2 of [10]) the general solution to the system of difference equations (2.1) associated with (1.1) can be written in the form

$$\varphi(s) = t_1^{s_1} \ldots t_n^{s_n} \frac{\prod_{i=1}^{p} \Gamma(\langle A_i, s\rangle - c_i)}{\prod_{i=1}^{n} \prod_{j=1}^{d_i} \Gamma((\Lambda s)_i - \alpha_{ij} + 1)} \phi(s), \tag{4.1}$$

where $p \in \mathbb{N}_0$, $t_i, c_i \in \mathbb{C}$, $A_i \in \mathbb{Z}^n$ and $\phi(s)$ is an arbitrary function satisfying the periodicity conditions $\phi(s+u_i) \equiv \phi(s)$, $i = 1, \ldots, n$. (Given polynomials P_i, Q_i satisfying the compatibility conditions (2.3), the parameters p, t_i, c_i, A_i of the solution $\varphi(s)$ can be computed explicitly. For a concrete construction of the function $\varphi(s)$ see [16]. The following Theorem holds true.

Theorem 4.1 *Suppose that the following conditions are fulfilled.*
1. For any $i, j = 1, \ldots, n$, $i \neq j$ it holds $Q_i(s+u_j) = Q_i(s)$ and $\deg Q_i > \deg P_i$.
2. The difference $\alpha_{ij} - \alpha_{ik}$ is never equal to a real integer number, for any $i = 1, \ldots, n$ and $j \neq k$.
3. For any multi-index $I = (i_1, \ldots, i_n)$ with $i_k \in \{1, \ldots, d_k\}$ the product $\prod_{i=1}^{p} (\langle A_i, s\rangle - c_i)$ never vanishes on the shifted lattice $\mathbb{Z}^n + \gamma_I$, where $\gamma_I = (\alpha_{1i_1}, \ldots, \alpha_{ni_n})$.

Then the family consisting of $\prod_{i=1}^{n} d_i$ functions

$$y_I(x) = x^{\gamma_I} \sum_{s \in \mathbb{Z}^n \cap K_U} t^{s+\gamma_I} \frac{\prod_{i=1}^{p} \Gamma(\langle A_i, s+\gamma_I\rangle - c_i)}{\prod_{k=1}^{n} \prod_{j=1}^{d_k} \Gamma((\Lambda s)_k + \alpha_{ki_k} - \alpha_{kj} + 1)} x^s \tag{4.2}$$

is a basis in the space of holomorphic solutions to the system (1.1) at any point $x \in (\mathbb{C}^*)^n = (\mathbb{C} \setminus \{0\})^n$. Here K_U is the cone spanned by the vectors $u_1, \ldots, u_n$.

Proof It follows from Theorem 2.1 and the assumptions 2,3 of Theorem 4.1 that the series (4.2) formally satisfies the generalized Horn system (1.1). Let χ_k denote the kth row of Λ. Since $\deg Q_i(s) > \deg P_i(s)$, $i = 1, \ldots, n$ it follows by the construction of the function (4.1) (see [16]) that all the components of the vector $\triangle = \sum_{i=1}^{p} A_i - \sum_{i=1}^{n} d_i \chi_i$ are negative. Thus for any multi-index I the intersection of the half-space $\mathrm{Re}\langle \triangle, s\rangle \geq 0$ with the shifted octant $K_U + \gamma_I$ is a bounded set. Using the Stirling formula we conclude that the series (4.2) converges everywhere in $(\mathbb{C}^*)^n$ for any multi-index I.

The series (4.2) corresponding to different multi-indices I, J are linearly independent since by the second assumption of Theorem 4.1 their initial monomials $x^{\gamma_I}, x^{\gamma_J}$ are different. Finally, the conditions of Theorem 3.3 are satisfied in our setting since the first assumption of Theorem 4.1 yields that the sequence of principal symbols $H_1(x^{(0)}, z), \ldots, H_n(x^{(0)}, z) \in R$ of hypergeometric differential operators defining the generalized Horn system is regular for $x^{(0)} \in (\mathbb{C}^*)^n$. Hence by Theorem 3.3 the number of linearly independent holomorphic solutions to the system under study at a generic point equals $\prod_{i=1}^n d_i$. In this case $U_{\mathcal{M}} = \{x^{(0)} \in \mathbb{C}^n : x_1^{(0)} \ldots x_n^{(0)} = 0\}$. Thus the series (4.2) span the space of holomorphic solutions to the system (1.1) at any point $x^{(0)} \in (\mathbb{C}^*)^n$. The proof is complete.

In the theory developed by Gelfand, Kapranov and Zelevinsky the conditions 2 and 3 of Theorem 4.1 correspond to the so-called nonresonant case (see [9], § 8.1). Thus the result on the structure of solutions to the generalized Horn system can be formulated as follows.

Corollary 4.2 *Let $x^{(0)} \in (\mathbb{C}^*)^n$ and suppose that $Q_i(s + u_j) = Q_i(s)$ and $\deg Q_i > \deg P_i$ for any $i, j = 1, \ldots, n$, $i \neq j$. If the parameters of the system (1.1) are nonresonant then there exists a basis in the space of holomorphic solutions to (1.1) near $x^{(0)}$ whose elements are given by series of the form (1.2).*

5 Examples

In this section we use the results on the structure of solutions to the generalized Horn system for computing the number of Laurent expansions of some rational functions. This problem is closely related to the notion of the amoeba of a Laurent polynomial, which was introduced by Gelfand et al. in [12] (see Chapter 6, § 1). Given a Laurent polynomial f, its amoeba $\mathcal{A}_f$ is defined to be the image of the hypersurface $f^{-1}(0)$ under the map $(x_1, \ldots, x_n) \mapsto (\log|x_1|, \ldots, \log|x_n|)$. This name is motivated by the typical shape of $\mathcal{A}_f$ with tentacle-like asymptotes going off to infinity. The connected components of the complement of the amoeba are convex and each such component corresponds to a specific Laurent series development with the center at the origin of the rational function $1/f$ (see [12], Chapter 6, Corollary 1.6). The problem of finding all such Laurent series expansions of a given Laurent polynomial was posed in [12] (Chapter 6, Remark 1.10).

Let $f(x_1, \ldots, x_n) = \sum_{\alpha \in S} a_\alpha x^\alpha$ be a Laurent polynomial. Here S is a finite subset of the integer lattice $\mathbb{Z}^n$ and each coefficient a_α is a non-zero complex number. The Newton polytope $\mathcal{N}_f$ of the polynomial f is defined to be the convex hull in $\mathbb{R}^n$ of the index set S. The following result was obtained in [8].

Theorem 5.1 *Let f be a Laurent polynomial. The number of Laurent series expansions with the center at the origin of the rational function $1/f$ is at least equal to the number of vertices of the Newton polytope $\mathcal{N}_f$ and at most equal to the number of integer points in $\mathcal{N}_f$.*

In the view of Corollary 1.6 in Chapter 6 of [12], Theorem 5.1 states that the number of connected components of the complement of the amoeba $\mathcal{A}_f$ is bounded from below by the number of vertices of $\mathcal{N}_f$ and from above by the number of integer points in $\mathcal{N}_f$. The lower bound has already been obtained in [12]. In this section we describe a class of rational functions for which the number of Laurent expansions attains the lower bound given by Theorem 5.1. Our main tool is Theorem 2.1 which allows one to describe supports of the Laurent series expansions of a rational function which can be treated as a solution to a generalized Horn system. In the following three examples we let $u_1, \ldots, u_n \in \mathbb{Z}^n$ be linearly independent vectors, $p \in \mathbb{N}$ and let $a_1, \ldots, a_n \in \mathbb{C}^*$ be nonzero complex numbers. We denote by U the matrix with the rows $u_1, \ldots, u_n$ and use the notation $(\lambda_{ij}) = \Lambda = (U^T)^{-1}$ and $\nu_i = \lambda_{1i} + \cdots + \lambda_{ni}$. The conclusions in all of the following examples can be deduced from Theorem 7 in [14].

Example 5.2 The function $y_1(x) = (1 - a_1 x^{u_1} - \cdots - a_n x^{u_n})^{-1}$ satisfies the following system of the Horn type

$$\begin{pmatrix} a_1 x^{u_1} \\ \cdots \\ a_n x^{u_n} \end{pmatrix} (\nu_1\theta_1 + \cdots + \nu_n\theta_n + 1)\, y(x) = \Lambda \begin{pmatrix} \theta_1 \\ \ldots \\ \theta_n \end{pmatrix} y(x). \tag{5.1}$$

Indeed, after the change of variables $x_i(\xi_1, \ldots, \xi_n) = \xi_1^{\lambda_{1i}} \ldots \xi_n^{\lambda_{ni}}$ (whose inverse is $\xi_i = x^{u_i}$) the system (5.1) takes the form

$$a_i \xi_i \left(\theta_{\xi_1} + \cdots + \theta_{\xi_n} + 1\right) y(\xi) = \theta_{\xi_i} y(\xi), \quad i = 1, \ldots, n. \tag{5.2}$$

The function $(1 - a_1\xi_1 - \cdots - a_n\xi_n)^{-1}$ satisfies (5.2) and therefore the function $y_1(x)$ is a solution of (5.1). The hypergeometric system (5.1) is a special instance of systems (5.3) and (5.5). We treat this simple case first in order to make the main idea more transparent.

By Theorem 3.3 the space of holomorphic solutions to (5.1) has dimension one at a generic point and hence $y_1(x)$ is the only solution to this system. Thus the supports of the Laurent series expansions of $y_1(x)$ can be found by means of Theorem 2.1. There exist $n+1$ subsets of the lattice $\mathbb{Z}^n$ which satisfy the conditions in Theorem 2.1 and can give rise to a Laurent expansion of $y_1(x)$ with nonempty domain of convergence. These subsets are $S_0 = \{s \in \mathbb{Z}^n : (\Lambda s)_i \geq 0,\ i = 1, \ldots, n\}$ and $S_j = \{s \in \mathbb{Z}^n : \nu_1 s_1 + \cdots + \nu_n s_n + 1 \leq 0,\ (\Lambda s)_i \geq 0,\ i \neq j\}$, $j = 1, \ldots, n$. Besides $S_0, \ldots, S_n$ there can exist other subsets of $\mathbb{Z}^n$ satisfying the conditions in Theorem 2.1. (Such subsets "penetrate" some of the hyperplanes $(\Lambda s)_i = 0$, $\nu_1 s_1 + \cdots + \nu_n s_n + 1 = 0$ without intersecting them; subsets of this type can only appear if $|\det U| \geq 1$). However, none of these additional subsets gives rise to a convergent Laurent series and therefore does not define an expansion of $y_1(x)$. Indeed, in any series with the support in a "penetrating" subset at least one index of summation necessarily runs from $-\infty$ to ∞. Letting all the variables, except for that one which corresponds to this index, be equal to zero, we obtain a hypergeometric series in one variable. The classical result on convergence of one-dimensional hypergeometric series (see [10], § 1) shows that this series is necessarily divergent. Thus the number of Laurent series developments of $y_1(x)$ cannot exceed $n+1$. The Newton polytope of the polynomial $1/y_1(x)$ has $n+1$ vertices since the vectors $u_1, \ldots, u_n$ are linearly independent. Using Theorem 5.1 we conclude that the number of Laurent series expansions of $y_1(x)$ equals $n+1$. Thus the lower bound for the number of connected components of the amoeba complement is attained.

Example 5.3 Recall that θ denotes the vector $\left(x_1\frac{\partial}{\partial x_1}, \ldots, x_n\frac{\partial}{\partial x_n}\right)$ and let $(\Lambda\theta)_i$ denote the ith component of the vector $\Lambda\theta$. Let $\mathcal{G}$ be the differential operator defined by

$$\mathcal{G} = (\Lambda\theta)_1 + \cdots + (\Lambda\theta)_{n-1} + p(\Lambda\theta)_n + p.$$

The function $y_2(x) = ((1 - a_1 x^{u_1} - \cdots - a_{n-1} x^{u_{n-1}})^p - a_n x^{u_n})^{-1}$ is a solution to the following system of differential equations of hypergeometric type

$$\left\{ \begin{array}{l} a_i x^{u_i} \mathcal{G} y(x) = (\Lambda\theta)_i y(x), \quad i = 1, \ldots, n-1, \\ a_n x^{u_n} \left(\prod_{j=0}^{p-1} (\mathcal{G} + j) \right) y(x) = \left(\prod_{j=0}^{p-1} \left(p(\Lambda\theta)_n + j\right) \right) y(x). \end{array} \right. \tag{5.3}$$

Indeed, the same monomial change of variables as in Example 5.2 reduces (5.3) to the system

$$\begin{cases} a_i\xi_i\,\tilde{\mathcal{G}}y(x) = \theta_{\xi_i}y(x), \;\; i = 1,\ldots,n-1, \\ a_n\xi_n\left(\prod_{j=0}^{p-1}(\tilde{\mathcal{G}}+j)\right)y(x) = \left(\prod_{j=0}^{p-1}(p\,\theta_{\xi_n}+j)\right)y(x), \end{cases} \tag{5.4}$$

where $\tilde{\mathcal{G}} = \theta_{\xi_1}+\cdots+\theta_{\xi_{n-1}}+p\theta_{\xi_n}+p$. The system (5.4) is satisfied by the function $((1-a_1\xi_1-\cdots-a_{n-1}\xi_{n-1})^p - a_n\xi_n)^{-1}$. This shows that $y_2(x)$ is indeed a solution to (5.3). Thus the support of a Laurent expansion of $y_2(x)$ must satisfy the conditions in Theorem 2.1. Notice that unlike (5.1), the system (5.3) can have solutions supported in subsets of the shifted lattice $\mathbb{Z}^n+\gamma$ for some $\gamma\in(0,1)^n$. Yet, such subsets are not of interest for us since we are looking for Laurent series developments of $y_2(x)$. The subsets $S_0 = \{s\in\mathbb{Z}^n : (\Lambda s)_i \geq 0,\ i=1,\ldots,n\}$ and $S_j = \{s\in\mathbb{Z}^n : (\Lambda s)_1+\cdots+(\Lambda s)_{n-1}+p(\Lambda s)_n+p \leq 0,\ (\Lambda s)_i\geq 0,\ i\neq j\}$, $j=1,\ldots,n$ satisfy the conditions in Theorem 2.1. The same arguments as in Example 5.2 show that no other subsets of $\mathbb{Z}^n$ satisfying the conditions in Theorem 2.1 can give rise to a convergent Laurent series which represents $y_2(x)$. This yields that the number of expansions of $y_2(x)$ is at most equal to $n+1$. The Newton polytope of the polynomial $1/y_2(x)$ has $n+1$ vertices since the vectors $u_1,\ldots,u_n$ are assumed to be linearly independent. Using Theorem 5.1 we conclude that the number of Laurent series developments of $y_2(x)$ equals $n+1$.

Example 5.4 Let $\mathcal{H}$ be the differential operator defined by $\mathcal{H} = p(\Lambda\theta)_2+\cdots+p(\Lambda\theta)_n+p$. Using the same change of variables as in Example 5.2, one checks that $y_3(x) = ((1-a_1x^{u_1})^p - a_2x^{u_2}-\cdots-a_nx^{u_n})^{-1}$ solves the system

$$\begin{cases} a_1x^{u_1}\left((\Lambda\theta)_1+\mathcal{H}\right)y(x) = (\Lambda\theta)_1\,y(x), \\ a_ix^{u_i}\frac{1}{p}\mathcal{H}\left(\prod_{j=0}^{p-1}((\Lambda\theta)_1+\mathcal{H}+j)\right)y(x) = \\ \qquad\qquad (\Lambda\theta)_i\left(\prod_{j=0}^{p-1}(\mathcal{H}-p+j)\right)y(x), \;\; i=2,\ldots,n. \end{cases} \tag{5.5}$$

Analogously to Example 5.2, we apply Theorem 2.1 to the system (5.5) and conclude that the number of Laurent expansions of $y_3(x)$ at most equals $n+1$. Thus it follows from Theorem 5.1 that the number of such expansions equals $n+1$.

Example 5.5 The Szegö kernel of the domain $\{z\in\mathbb{C}^2 : |z_1|+|z_2|<1\}$ is given by the hypergeometric series

$$h(x_1,x_2) = \sum_{s_1,s_2\geq 0}\frac{\Gamma(2s_1+2s_2+2)}{\Gamma(2s_1+1)\Gamma(2s_2+1)}x_1^{s_1}x_2^{s_2} = \frac{(1-x_1-x_2)(1+2x_1x_2-x_1^2-x_2^2)+8x_1x_2}{\left((1-x_1-x_2)^2-4x_1x_2\right)^2}. \tag{5.6}$$

(See [2], Chapter 3, § 14.) This series satisfies the system of equations

$$x_i\,(2\theta_1+2\theta_2+3)\,(2\theta_1+2\theta_2+2)\,y(x) = 2\theta_i(2\theta_i-1)y(x), \;\; i=1,2.$$

There exist three subsets of the lattice $\mathbb{Z}^n$ which satisfy the conditions in Theorem 2.1, namely $\{s \in \mathbb{Z}^2 : s_1 \geq 0, s_2 \geq 0\}$, $\{s \in \mathbb{Z}^2 : s_1 \geq 0, s_1 + s_2 + 1 \leq 0\}$, $\{s \in \mathbb{Z}^2 : s_2 \geq 0, s_1 + s_2 + 1 \leq 0\}$. Using Theorem 2.1 we conclude that the number of Laurent expansions centered at the origin of the Szegö kernel (5.6) at most equals 3. The Newton polytope of the denominator of the rational function (5.6) is the simplex with the vertices $(0,0)$, $(4,0)$, $(0,4)$. By Theorem 5.1 the number of Laurent series developments of the Szegö kernel at least equals 3. Thus the number of Laurent expansions of (5.6) (or, equivalently, the number of connected components in the complement of the amoeba of its denominator) attains its lower bound.

Example 5.6 Let $u_1 = (1,0), u_2 = (1,1)$ and consider the system of equations

$$\begin{cases} x^{u_1} y(x) &= \left(x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2}\right) y(x), \\ x^{u_2} y(x) &= \left(x_2 \frac{\partial}{\partial x_2}\right) y(x). \end{cases} \tag{5.7}$$

The principal symbols $H_1(x,z), H_2(x,z) \in R[x]$ of the differential operators defining the system (5.7) are given by $H_1(x,z) = -x_1 z_1 + x_2 z_2$, $H_2(x,z) = -x_2 z_2$. By Theorem 3.3 the dimension of the solution space of (5.7) at a generic point is equal to 1 since $\dim_{\mathbb{C}} R\big/(H_1(x,z), H_2(x,z)) = 1$ for $x_1 x_2 \neq 0$. For computing the solution to (5.7) explicitly we choose $\gamma = 0$ and consider the corresponding system of difference equations

$$\begin{cases} \varphi(s+u_1)(s_1 - s_2 + 1) &= \varphi(s), \\ \varphi(s+u_2)(s_2+1) &= \varphi(s). \end{cases} \tag{5.8}$$

The general solution to (5.8) is given by $\varphi(s) = \left(\Gamma(s_1 - s_2 + 1)\Gamma(s_2+1)\right)^{-1}\phi(s)$, where $\phi(s)$ is an arbitrary function which is periodic with respect to the vectors u_1, u_2.

There exists only one subset of $\mathbb{Z}^2$ satisfying the conditions of Theorem 2.1, namely $S = \{(s_1, s_2) \in \mathbb{Z}^2 : s_1 - s_2 \geq 0,\ s_2 \geq 0\}$. Choosing $\phi(s) \equiv 1$ and using (4.2), we obtain the solution to (5.7):

$$y(x) = \sum_{\substack{s_1 - s_2 \geq 0, \\ s_2 \geq 0}} \frac{x_1^{s_1} x_2^{s_2}}{\Gamma(s_1 - s_2 + 1)\Gamma(s_2+1)} = \exp(x_1 x_2 + x_1). \tag{5.9}$$

It is straightforward to check that the solution space of (5.7) is indeed spanned by (5.9).

Bibliography

1. A. Adolphson. Hypergeometric functions and rings generated by monomials, Duke Math. J. 73 (1994), 269-290.

2. L. Aizenberg. Carleman's Formulas in Complex Analysis. Theory and Applications, Kluwer Academic Publishers, 1993.

3. J.-E. Björk. Rings of Differential Operators, North. Holland Mathematical Library, 1979.

4. J.-E. Björk. Analytic $\mathcal{D}$-Modules and Applications, Kluwer Academic Publishers, 1993.

5. E. Cattani, C. D'Andrea, A. Dickenstein. The $\mathcal{A}$-hypergeometric system associated with a monomial curve, Duke Math. J. 99 (1999), 179-207.

6. E. Cattani, A. Dickenstein, B. Sturmfels. Rational hypergeometric functions, Compos. Math. 128 (2001), 217-240.

7. A. Dickenstein, T. Sadykov. Bases in the solution space of the Mellin system, math.AG/0609675, to appear in Sbornik Mathematics.

8. M. Forsberg, M. Passare, A. Tsikh. Laurent determinants and arrangements of hyperplane amoebas, Adv. Math. 151 (2000), 45-70.

9. I.M. Gelfand, M.I. Graev. GG-functions and their relation to general hypergeometric functions, Russian Math. Surveys 52 (1997), 639-684.

10. I.M. Gelfand, M.I. Graev, V.S. Retach. General hypergeometric systems of equations and series of hypergeometric type, Russian Math. Surveys 47 (1992), 1-88.

11. I.M. Gelfand, M.I. Graev, V.S. Retach. General gamma functions, exponentials, and hypergeometric functions, Russian Math. Surveys 53 (1998), 1-55.

12. I.M. Gelfand, M.M. Kapranov, A.V. Zelevinsky. Discriminants, Resultants and Multidimensional Determinants, Birkhäuser, Boston, 1994.

13. J. Horn. Über hypergeometrische Funktionen zweier Veränderlicher, Math. Ann. 117 (1940), 384-414.

14. M. Passare, T.M. Sadykov, A.K. Tsikh. Nonconfluent hypergeometric functions in several variables and their singularities, Compos. Math. 141 (2005), no. 3, 787–810.

15. M. Passare, A. Tsikh, O. Zhdanov. A multidimensional Jordan residue lemma with an application to Mellin-Barnes integrals, Aspects Math. E 26 (1994), 233-241.

16. T.M. Sadykov. On the Horn system of partial differential equations and series of hypergeometric type, Math. Scand. 91 (2002), 127-149.

17. M. Sato. Singular orbits of a prehomogeneous vector space and hypergeometric functions, Nagoya Math. J. 120 (1990), 1-34.

18. A.K. Tsikh. Multidimensional Residues and Their Applications, Translations of Mathematical Monographs, 103. American Mathematical Society, Providence, 1992.

РАЗРЯЖЕННЫЕ ГИПЕРГЕОМЕТРИЧЕСКИЕ СИСТЕМЫ

Тимур Садыков
Сибирский федеральный университет,
пр. Свободный, 79, Красноярск, 660041, Россия, e-mail: sadykov@lan.krasu.ru

Аннотация. Описывается подход к изучению теории гипергеометрических функций от нескольких переменных с помощью обобщенной системы дифференциальных уравнений типа Горна. Получена формула для вычисления размерности пространства решений этой системы, основываясь на которой строится в явном виде базис ее пространства голоморфных решений при некоторых ограничениях на параметры системы.

Ключевые слова: гипергеометрические функции, системы дифференциальных уравнений типа Горна, система Меллина.

УДК 517.55

REFLECTION GROUPS OF RANK THREE AND SYSTEMS OF UNIFORMIZATION EQUATIONS

Jiro Sekiguchi

Department of Mathematics,Faculty of Engineering, Tokyo University of Agriculture and Technology, Koganei, Tokyo, 184-8588, Japan, e-mail: sekiguti@cc.tuat.ac.jp

Abstract. In this note, I survey recent progress on systems of uniformization equations along Saito free divisors. In particular, I show concrete forms of such systems along Saito free divisors which are defined as the zero sets of discriminants of complex reflection groups of rank three. A part of the results in this note is a joint work with M. Kato (Univ. Ryukyus)

Keywords: uniformization equations, discriminants of complex reflection groups, Saito free divisors.

Introduction

The notion of Saito free divisors was introduced by K. Saito (cf. [10]). He also formulated and stressed the importance of systems of uniformization equations along such divisors (cf. [9]). In this note, I explain recent progress on systems of uniformization equations along Saito free divisors defined as the zero sets of the discriminants of complex reflection groups of rank three. The hypersurface defined by the discriminant of a real reflection group is a typical example of Saito free divisors. It is known that the discriminant of a complex reflection group of rank three is also a Saito free divisor (cf. [8], [5]). But I don't know whether it is true for the case of arbitrary complex reflection groups. My interests on this subject are to construct (1) Saito free divisors, (2) systems of uniformization equations, and (3) their solutions in a concrete manner. Restricting to the case of three dimensional affine space, I obtained some results on (1), (2). But it is difficult to attack (3) compared with (1), (2). The purpose of this note is to report my results on (1), (2) for the discriminants of irreducible complex reflection groups of rank three. A part of the results of the last three sections are obtained by a joint research with M. Kato (Univ. Ryukyus).

1 Definition of Saito free divisors

Let $F(x) = F(x_1, x_2, \dots, x_n)$ be a reduced polynomial. Then $D = \{x \in \mathbf{C}^n;\ F(x) = 0\}$ is a (weighted homogeneous) Saito free divisor if (C1)+(C2) hold.

(C1) There is a vector field

$$E = \sum_{i=1}^{n} m_i x_i \partial_{x_i}$$

such that $EF = dF$, where $m_1, m_2, \dots, m_n, d$ are positive integers with $0 < m_1 \le m_2 \le \dots \le m_n$.

(C2) There are vector fields

$$V^i = \sum_{j=1}^{n} a_{ij}(x) \partial_{x_j}\ (i = 1, 2, \dots, n)$$

such that

(i) each $a_{ij}(x)$ is a polynomial of $x_1, x_2, \dots, x_n$,

(ii) $\det(a_{ij}(x)) = cF(x)$ for a non-zero constant c,

(iii) $V^1 = E$, $V^i F(x) = c_i(x)F(x)$ for polynomials $c_i(x)$,

(iv) $[E, V^i] = k_i V^i$ for some constants k_i,

(v) V^i $(j = 1, 2, \dots, n)$ form a Lie algebra over $R = \mathbf{C}[x_1, x_2, \dots, x_n]$

We now give examples of Saito free divisors.

Let

$$f(t) = t^n + x_2 t^{n-2} + x_3 t^{n-3} + \cdots + x_{n-1} t + x_n$$

be a polynomial of nth degree and let $\Delta(x_2, x_3, \dots, x_n)$ be the discriminant of $f(t)$. Then $\Delta = 0$ is a Saito free divisor in $\mathbf{C}^{n-1}$.

More generally, the zero locus of the discriminant of an irreducible real reflection group is a Saito free divisor.

Basic reference of this section is [10].

2 Irreducible complex reflection groups of rank three.

In this section, we collect some results on irreducible complex reflection groups of rank three. A basic reference on complex reflection groups is Shephard-Todd [16] (see also [8]).

Reflection groups treated in this section are real reflection groups of types A_3, B_3, H_3 and complex reflection groups of No.24,No.25,No.26,No.27 in the sense of [16]. The real reflection group of type H_3 is same as the group No. 23 in [16].

Let G be one of the seven groups and let P_1, P_2, P_3 algebraically independent basic G-invariant polynomials and put $k_j = \deg_\xi(P_j)$. We may assume that $k_1 \le k_2 \le k_3$. Let r be the greatest common divisor of k_1, k_2, k_3 and put $k'_j = k_j/r$ $(j = 1, 2, 3)$. For the later convenience, we write x_1, x_2, x_3 for P_1, P_2, P_3. Let $\delta_G(x_1, x_2, x_3)$ be the discriminant of G expressed as a polynomial of x_1, x_2, x_3.

In the cases A_3, B_3, H_3, taking G-invariants x_1, x_2, x_3 suitably, $F_{W(A_3)}(x_1, x_2, x_3)$, $F_{W(B_3)}(x_1, x_2, x_3)$, $F_{W(H_3)}(x_1, x_2, x_3)$ are discriminants for G up to a constant factor, respectively, where $F_{A,1}, F_{B,1}, F_{H,1}$ are the polynomials given in Theorem of [13].

	group	order	k_1, k_2, k_3	degree	(k'_1, k'_2, k'_3)
A_3	$W(A_3)$	24	$2, 3, 4$	12	$(2, 3, 4)$
B_3	$W(B_3)$	48	$2, 4, 6$	18	$(1, 2, 3)$
H_3	$W(H_3)$	120	$2, 6, 10$	30	$(1, 3, 5)$
No.24	G_{336}	336	$4, 6, 14$	42	$(2, 3, 7)$
No.25	G_{648}	648	$6, 9, 12$	36	$(2, 3, 4)$
No.26	G_{1296}	1296	$6, 12, 18$	54	$(1, 2, 3)$
No.27	G_{2160}	2160	$6, 12, 30$	90	$(1, 2, 5)$

The concrete forms of discriminants of $W(A_3)$, $W(B_3)$, $W(H_3)$ are as follows:

Type A_3: $16x_1^4x_3 - 4x_1^3x_2^2 - 128x_1^2x_3^2 + 144x_1x_2^2x_3 - 27x_2^4 + 256x_3^3$.

Type B_3: $x_3(x_1^2x_2^2 - 4x_2^3 - 4x_1^3x_3 + 18x_1x_2x_3 - 27x_3^2)$.

Type H_3: $-50x_3^3 + (4x_1^5 - 50x_1^2x_2)x_3^2 + (4x_1^7x_2 + 60x_1^4x_2^2 + 225x_1x_2^3)x_3 - \frac{135}{2}x_2^5$
$-115x_1^3x_2^4 - 10x_1^6x_2^3 - 4x_1^9x_2^2$.

The discriminants for the groups No.25, No.26 are same as those for $W(A_3)$, $W(B_3)$ respectively by taking the basic invariants suitably. On the other hand, those for the groups No.24, No.27 will be given in §5, §.6.

3 Systems of uniformization equations along Saito free divisors.

Let $F(x) = 0$ be a Saito free divisor and let V^j $(j = 1, 2, \dots, n)$ be basic vector fields logarithmic along $F = 0$. Let $u = u(x_1, x_2, \dots, x_n)$ be an unknown function. Assume that $V^1 u = su$ for a constant $s(\neq 0)$ Put $\vec{u} = {}^t(u, V^2 u, \dots, V^n u)$ and consider the system of differential equations

(UE) $V^j \vec{u} = A_j(x)\vec{u} \quad (j = 1, 2, \dots, n)$

where $A_j(x)$ $(j = 1, 2, \dots, n)$ are $n \times n$ matrices whose entries are polynomials of x.

If (UE) is integrable, it is called a system of uniformization equations with respect to the Saito free divisor $F = 0$.

The system (UE) is written by

$$\text{(UEa)} \begin{cases} V^1 u = su \\ V^i V^j u = \displaystyle\sum_{k=1}^{n} h_{ij}^k(x) V^k u \quad (\forall i, j) \end{cases}$$

where $h_{ij}^k(x)$ are polynomials of x. There are n number of fundamental solutions of (UEa). Let $u_j(x)$ $(j = 1, 2, \dots, n)$ be fundamental solutions outside the divisor $F = 0$. Then

$$\varphi(x) = (u_1(x), u_2(x), \dots, u_n(x))$$

defines a map of $\mathbf{C}^n - \{F = 0\}$ to $\mathbf{C}^n$. The following two problems are fundamental in the study on systems of uniformization equations.

PROBLEM 1: Construct fundamental solutions $u_j(x)$ $(j = 1, 2, \dots, n)$ of (UEa).

PROBLEM 2: Construct the inverse of $\varphi(x)$ in a concrete manner.

These two problems are solved in the case $W(A_3)$ for a special but interesting system of uniformization equations by K. Saito. For the details of the results, see [9].

4 The discriminant of the Coxeter group $W(H_3)$ of type H_3.

A part of the argument in the case of $W(A_3)$ in [9] is applicable to the case of the Saito free divisor defined by the zero locus of the discriminant of the Coxeter group of type H_3. In this section, I will explain the results on this case. For the details of results in this section, see [15].

The discriminant of the polynomial $P(t)$ defined by

$$P(t) = t^6 + y_1 t^5 + y_2 t^3 + y_3 t + \frac{1}{20} y_2^2 - \frac{1}{4} y_1 y_3 \tag{4.1}$$

is Δ^2 up to a constant factor, where

$$\Delta = 125 y_1^3 y_2^4 + 864 y_2^5 - 1250 y_1^4 y_2^2 y_3 - 9000 y_1 y_2^3 y_3 + 3125 y_1^5 y_3^2 + 25000 y_1^2 y_2 y_3^2 + 50000 y_3^3. \tag{4.2}$$

Remark 4.1 *The equation $P(t) = 0$ is essentially same as "Die allgemeine Jacobi'sche Gleichung sechsten Grades" (see p.223 in Klein's book [7]).*

The polynomial Δ is regarded as the discriminant of the group $W(H_3)$. In fact, the substitution of the variables (y_1, y_2, y_3) with (x_1, x_2, x_3) defined by the relations

$$\left\{ \begin{array}{lcl} y_1 & = & -4x_1 \\ y_2 & = & 10x_1^3 - 25x_2 \\ y_3 & = & -4x_1^5 + 50x_1^2 x_2 - 50x_3 \end{array} \right. \tag{4.3}$$

implies that Δ coincides with the determinant of the matrix M up to a constant factor, where M is defined by

$$M = \begin{pmatrix} x_1 & 3x_2 & 5x_3 \\ 3x_2 & 2x_3 + 2x_1^2 x_2 & 7x_1 x_2^2 + 2x_1^4 x_2 \\ 5x_3 & 7x_1 x_2^2 + 2x_1^4 x_2 & \frac{1}{2}(15x_2^3 + 4x_1^4 x_3 + 18x_1^3 x_2^2) \end{pmatrix} \tag{4.4}$$

and $\det M$ is the discriminant of $W(H_3)$ (cf. [17]). In the sequel, we always regard $P(t)$ as a polynomial of t and x. The hypersurface defined as the zero set of the polynomial $f_0 = \det M$ is an example of Saito free divisors. To show this, we define vector fields V_0, V_1, V_2 by

$$\begin{pmatrix} V_0 \\ V_1 \\ V_2 \end{pmatrix} = M \begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \\ \partial_{x_3} \end{pmatrix}$$

Then we have

$$[V_0, V_1] = 2V_1,\ [V_0, V_2] = 4V_2,$$

$$[V_1, V_2] = (4x_1^3 x_2 + 2x_2^2)V_0 + 4x_1 x_2 V_1$$

and

$$\begin{array}{l} V_0 f_0 = 15 f_0, \\ V_1 f_0 = 2x_1^2 f_0, \\ V_2 f_0 = 2x_1(2x_1^3 + 5x_2) f_0 \end{array}$$

Remark 4.2 *We note that*

$$P(-x_1) = \frac{125}{4} x_2^2. \tag{4.5}$$

This implies that $(-x_1, 5\sqrt{5}/2 \cdot x_2)$ is a point on the hyperelliptic curve $s^2 = P(t)$ on (s, t) plane.

Consider the system of differential equations

$$V_i \begin{pmatrix} u \\ V_1 u \\ V_2 u \end{pmatrix} = B_{i+1} \begin{pmatrix} u \\ V_1 u \\ V_2 u \end{pmatrix} \quad (i = 0, 1, 2) \tag{4.6}$$

The system (4.6) is a system of uniformization equations along the Saito free divisor $f_0(x) = 0$. Here B_j are defined as follows:

$$B_1 = \begin{pmatrix} s_0 & 0 & 0 \\ 0 & 2+s_0 & 0 \\ 0 & 0 & 4+s_0 \end{pmatrix}$$

$$B_2 = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{2}{225}x_1 \left\{ \begin{array}{c} (8+70s_1-100s_1^2+8s_0 \\ +35s_1s_0+2s_0^2)x_1^3 \\ +(-180+825s_1-750s_1^2 \\ -90s_0+75s_1s_0)x_2 \end{array} \right\} & \frac{1}{15}(8+5s_1+4s_0)x_1^2 & s_1 \\ \frac{1}{900}\left\{ \begin{array}{c} (-128+80s_1+100s_1^2-128s_0 \\ +40s_1s_0-32s_0^2)x_1^6 \\ +10(-32-400s_1+550s_1^2+328s_0 \\ -20s_1s_0-8s_0^2)x_1^3x_2 \\ +(-4500s_1+5625s_1^2+1800s_0)x_2^2 \\ +(2400-3000s_1+1200s_0)x_1x_3 \end{array} \right\} & \frac{1}{15}x_1\left\{ \begin{array}{c} (8+5s_1+4s_0)x_1^3 \\ +10(8+5s_1+s_0)x_2 \end{array} \right\} & \frac{1}{15}(4-5s_1+2s_0)x_1^2 \end{pmatrix}$$

$$B_3 = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{900}\left\{ \begin{array}{c} (-128+80s_1+100s_1^2-128s_0 \\ +40s_1s_0-32s_0^2)x_1^6 \\ +10(-32-400s_1+550s_1^2-32s_0 \\ -20s_1s_0-8s_0^2)x_1^3x_2 \\ +s_1(-4500+5625s_1)x_2^2 \\ +100(24-30s_1+12s_0)x_1x_3 \end{array} \right\} & \frac{1}{15}x_1\left\{ \begin{array}{c} (8+5s_1+4s_0)x_1^3 \\ +10(2+5s_1+s_0)x_2 \end{array} \right\} & \frac{1}{15}(4-5s_1+2s_0)x_1^2 \\ \frac{1}{450}\left\{ \begin{array}{c} (-128+80s_1+100s_1^2-128s_0 \\ +40s_1s_0-32s_0^2)x_1^8 \\ +(80-500s_1+500s_1^2-280s_0 \\ +200s_1s_0-160s_0^2)x_1^5x_2 \\ +25(-104-130s_1+325s_1^2+40s_0 \\ +25s_1s_0-8s_0^2)x_1^2x_2^2 \\ +100(12-15s_1+24s_0)x_1^3x_3 \\ +50(60-75s_1+30s_0)x_2x_3 \end{array} \right\} & \frac{1}{4}(4+5s_1)x_2(4x_1^3+5x_2) & \frac{x_1}{15}\left\{ \begin{array}{c} (16-5s_1+8s_0)x_1^3 \\ +(40-50s_1+20s_0)x_2 \end{array} \right\} \end{pmatrix}$$

Remark 4.3 *In the case* $s_0 = \frac{1}{2}$, $s_1 = 1$, *the monodromy group of the system* $V_j\vec{u} = B_{j+1}\vec{u}(j = 0,1,2)$ *coincides with* $W(H_3)$. *This case is treated in Haraoka-Kato [6].*

We consider the system $V_j\vec{u} = B_{j+1}\vec{u}$ $(j = 0,1,2)$ with $s_0 = -2,\ s_1 = 0$. Then we obtain

$$\left\{ \begin{array}{lcl} V_0 v & = & -2v \\ V_1V_1v & = & 0 \\ V_2V_1v & = & 0 \\ V_2V_2v & = & -4x_1^2(3x_2^2+2x_1x_3)v + x_2(4x_1^3+5x_2)V_1v \end{array} \right. \tag{4.7}$$

Theorem 4.4 *(cf. [15]) The function* $v(x)$ *defined by*

$$v(x) = \int_{\infty}^{-x_1} P(t)^{-1/2}dt$$

is a solution of (4.7).

The proof of this theorem is given by an argument similar to the case of type A_3.

If $u(x)$ is a solution of (4.7) such that $V_1 u = 0$, then u is a solution of

$$\begin{cases} V_0 u = -2u \\ V_1 u = 0 \\ \{V_2^2 + 4x_1^2(3x_2^2 + 2x_1x_3)\}u = 0 \end{cases} \tag{4.8}$$

Taking two paths C_1, C_2 appropriately and define

$$w_j(x) = \int_{C_j} \varphi_1(t)dt \quad (j = 1, 2)$$

Then each $w_j(x)$ is also a solution of (4.7) and in this manner we can construct solutions of (4.8). In this case it is not clear whether solutions of (4.8) are expressed by special functions or not. Moreover **PROBLEM 2** (the construction of the inverse mapping) is still open.

5 The discriminant of the group G_{336}, Shephard-Todd notation No.24.

In this case, we begin with defining the polynomial

$$\begin{aligned} P(t) &= t^7 - \tfrac{7}{2}(c_1-1)x_2t^5 - \tfrac{7}{2}(c_1-1)x_3t^4 - 7(c_1+4)x_2^2t^3 - 14(c_1+2)x_2x_3t^2 \\ &\quad + \tfrac{7}{2}\{(3c_1-7)x_2^3 - (c_1+5)x_3^2\}t + \tfrac{1}{2}(7c_1-131)x_2^2x_3 + x_7 \\ &\quad (c_1^2 = -7) \end{aligned}$$

The discriminant of $P(t)$ is f_0^2 up to a constant factor, where

$$\begin{aligned} f_0 &= 2048x_2^9x_3 - 22016x_2^6x_3^3 + 60032x_2^3x_3^5 - 1728x_3^7 + 256x_2^7x_7 - 1088x_2^4x_3^2x_7 \\ &\quad - 1008x_2x_3^4x_7 + 88x_2^2x_3x_7^2 - x_7^3 \end{aligned}$$

and f_0 is the discriminant of the complex reflection group G_{336}. (The polynomial f_0 is same as the one shown in p.262 of the paper of A. Adler in the book "*The Eightfold Way*"by $x_2 \to f$, $x_3 \to \nabla$, $x_7 \to C$. The polynomial $P(t)$ is given in p.406 of GMA of F. Klein, Band II.)

Define vector fields V_0, V_1, V_2 by

$${}^t(V_0, V_1, V_2) = M^t(\partial_{x_2}, \partial_{x_3}, \partial_{x_7})$$

Then V_0, V_1, V_2 form the generators of logarithmic vector fields along $f_0 = 0$. Here

$$M = \begin{pmatrix} 2x_2 & 3x_3 & 7x_7 \\ x_3^2 & -\frac{1}{12}x_7 & -\frac{4}{3}x_2(28x_2^3x_3 - 128x_3^3 + 3x_2x_7) \\ 7x_7 & -56x_2(2x_2^3 - 13x_3^2) & 28(32x_2^6 - 40x_2^3x_3^2 - 84x_3^4 + 59x_2x_3x_7) \end{pmatrix}$$

Put

```
A0={{s0,0,0},{0,s0+4,0},{0,0,s0+5}};
A1={{0,1,0},{1/162*x2*(4*(-1+c4-s0)*(8+c4+2*s0)*x2^3-
    3*(24+43*c4+5*c4^2+24*s0+
     19*c4*s0)*x3^2),1/9*(-10+c4-4*s0)*x2^2,c4*x3/504},
```

```
    {-7/54*(8*(-152+37*c4+7*c4^2-172*s0-14*c4*s0-38*s0^2)*x2^3*x3-
    18*(8*c4+c4^2+76*s0)*x3^3+3*(8+c4+38*s0)*x2*x7),
    -14/3*(-20+5*c4-38*s0)*x2*x3,-1/9*(8+c4+2*s0)*x2^2}};
A2={{0,0,1},
    {-7/54*(8*(-152+37*c4+7*c4^2-190*s0-14*c4*s0-38*s0^2)*x2^3*x3-18*c4*(8+
     c4)*x3^3+3*(8+c4+8*s0)*x2*x7),-14/3*(-152+5*c4-38*s0)*x2*x3,
     -1/9*(2+c4+2*s0)*x2^2},
    {98/9*(48*(-24+5*c4+c4^2-36*s0-c4*s0)*x2^5+4*(-440+97*c4+19*c4^2-658*s0
    -89*c4*s0-722*s0^2)*x2^2*x3^2+3*(8+c4+38*s0)*x3*x7),
    -1176*(-2+c4)*(2*x2^3-x3^2),14/3*(190+5*c4+76*s0)*x2*x3}};
```

There is a system of differential equation of rank three defined by

$$V_j \begin{pmatrix} u \\ V_1 u \\ V_2 u \end{pmatrix} = A_j \begin{pmatrix} u \\ V_1 u \\ V_2 u \end{pmatrix} \quad (j = 0, 1, 2)$$

This system has two parameters s_0, c_4.

Substituting $s_0 = -1, c_4 = 0$ in A_j, we obtain $A_j^{(0)}$;

$$A_0^{(0)} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \quad A_1^{(0)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -\frac{2}{3}x_2^2 & 0 \\ \frac{7}{3}(8x_2^3x_3 - 76x_3^3 + 5x_2x_7) & -84x_2x_3 & -\frac{2}{3}x_2^2 \end{pmatrix},$$

$$A_2^{(0)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 532x_2x_3 & 0 \\ 196(32x_2^5 - 112x_2^2x_3^2 - 5x_3x_7) & 2352(2x_2^3 - x_3^2) & 532x_2x_3 \end{pmatrix}$$

The system

$$V_j \begin{pmatrix} u \\ V_1 u \\ V_2 u \end{pmatrix} = A_j^{(0)} \begin{pmatrix} u \\ V_1 u \\ V_2 u \end{pmatrix} \quad (j = 0, 1, 2)$$

has a quotient which is defined by $V_1 u = 0$. Assuming $V_1 u = 0$, the system for $\begin{pmatrix} u \\ V_2 u \end{pmatrix}$ turns out to be

$$\begin{cases} V_0 \begin{pmatrix} u \\ V_2 u \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} u \\ V_2 u \end{pmatrix} \\ V_1 \begin{pmatrix} u \\ V_2 u \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \frac{7}{3}(8x_2^3x_3 - 76x_3^3 + 5x_2x_7) & -\frac{2}{3}x_2^2 \end{pmatrix} \begin{pmatrix} u \\ V_2 u \end{pmatrix} \\ V_2 \begin{pmatrix} u \\ V_2 u \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 196(32x_2^5 - 112x_2^2x_3^2 - 5x_3x_7) & 532x_2x_3 \end{pmatrix} \begin{pmatrix} u \\ V_2 u \end{pmatrix} \end{cases} \tag{5.1}$$

We now study the restriction of the system (5.1) to the hyperplane $x_2 = 0$. Then we obtain an ordinary differential equation

$$\left(\partial_{x_7}^2 + \frac{18x_7^2}{7(1728x_3^7 + x_7^3)} \partial_{x_7} + \frac{10x_7}{49(1728x_3^7 + x_7^3)} \right) u = 0$$

One of its solutions is

$$x_3^{-1/3}F\left(\frac{1}{21},\frac{10}{21};\frac{2}{3};-\frac{x_7^3}{1728x_3^7}\right)$$

Similarly as the restriction to $x_3=0$ of (5.1), we obtain an ordinary differential equation

$$\left(\partial_{x_7}^2-\frac{256x_2^7+11x_7^2}{7x_7(256x_2^7-x_7^2)}\partial_{x_7}+\frac{3}{49(-256x_2^7+x_7^2)}\right)u=0$$

One of its solutions is

$$x_2^{-1/2}F\left(\frac{1}{14},\frac{3}{14};\frac{3}{7};\frac{x_7^2}{256x_2^7}\right)$$

Remark 5.1 *We note that, in the case $c_4=-9$, $s_0=1/2$, the system of differential equations has a monodromy group isomorphic to G_{336}. This case is treated in [6].*

6 The discriminant of the group G_{2160}, Shephard-Todd notation No.27.

Consider the polynomial

$$P(t)=t^6+y_1t^5+y_2t^4+y_3t^3+y_4t^2+y_5t+y_6.$$

Substitute y_j $(j=1,2,\dots,6)$ by x_j $(j=1,2,\dots,6)$;

```
y1=x1
y2=(5/16)*(9 + sr)*x2,
y3=(5/64)*(11 + 3*sr)*x1*x2,
y4=(5/512)*(37 + 45*sr)*x2^2,
y5=(61 + 5*sr)*(-64*x1^3*x2 + 373*x1*x2^2 + 15*sr*x1*x2^2 + 2*x3))/12288,
y6=(-279 + 145*sr)*(-512*x1^4*x2 + 2864*x1^2*x2^2 +
 1425*x2^3 + 135*sr*x2^3 +16*x1*x3)/3538944,
```

where $sr^2=-15$. Then f_0^2 the discriminant of the polynomial $P(t)$, where

$$\begin{aligned} f_0 \;=\;& 65536x_1^{11}x_2^2-1765376x_1^9x_2^3+17406016x_1^7x_2^4-73887360x_1^5x_2^5+107371008x_1^3x_2^6\\ &+34338816x_1x_2^7-4096x_1^8x_2x_3+96640x_1^6x_2^2x_3-707952x_1^4x_2^3x_3+1622592x_1^2x_2^4x_3\\ &+186624x_2^5x_3+64x_1^5x_3^2-1584x_1^3x_2x_3^2+7128x_1x_2^2x_3^2+9x_3^3 \end{aligned}$$

up to a constant factor.

Remark 6.1 *By direct computation, we find that*

$$P\left(\frac{(3-5sr)}{72}x_1\right)=\frac{5(-45+11sr)}{1152}\left\{x_2-\frac{(39-sr)}{216}x_1^2\right\}^3$$

This means that

$$\left(\frac{(3-5sr)}{72}x_1,\left(\frac{5(-45+11sr)}{1152}\right)^{1/3}\left\{x_2-\frac{(39-sr)}{216}x_1^2\right\}\right)$$

is a point on the trielliptic curve $s^3=P(t)$ on (s,t) plane.

The polynomial f_0 is regarded as the discriminant of the complex reflection group No.27. In particular, f_0 is obtained as the determinant of the matrix

$$M = \begin{pmatrix} x_1 & 2x_2 & 5x_3 \\ x_2^2 & \frac{1}{432}(144x_1x_2^2 - x_3) & \begin{array}{c} \frac{1}{108}(640x_1^6x_2 - 9388x_1^4x_2^2 + 36600x_1^2x_2^3 \\ -19872x_2^4 - 28x_1^3x_3 + 307x_1x_2x_3) \end{array} \\ x_3 & \begin{array}{c} \frac{1}{135}(-1920x_1^4x_2 + 8724x_1^2x_2^2 \\ +16416x_2^3 + 139x_1x_3) \end{array} & \begin{array}{c} -\frac{4}{135}x_1(65920x_1^6x_2 - 887092x_1^4x_2^2 + 2886120x_1^2x_2^3 \\ +367632x_2^4 - 2692x_1^3x_3 + 20533x_1x_2x_3) \end{array} \end{pmatrix}.$$

It can be shown that f_0 coincides with the polynomial `gk13` in my notation by a weight preserving coordinate change in the notation of my note. We define vector fields V_0, V_1, V_2 by

$$\begin{pmatrix} V_0 \\ V_1 \\ V_2 \end{pmatrix} = M \begin{pmatrix} \partial_{x_1} \\ \partial_{x_2} \\ \partial_{x_3} \end{pmatrix}$$

Then V_0, V_1, V_2 form generators of the logarithmic vector fields along the set $f_0 = 0$ in the (x_1, x_2, x_3)-space. By direct computation, we have

$$[V_1, V_2] = \frac{1}{540}(3200x_1^5x_2 - 16412x_1^3x_2^2 - 18056x_1x_2^3 - 80x_1^2x_3 - 307x_2x_3)V_0$$

$$-\frac{8}{135}(474x_1^4 - 4102x_1^2x_2 + 7209x_2^2)V_1 - \frac{1}{54}x_1(6x_1^2 - 73x_2)V_2$$

We consider the system of differential equations

$$V_j \begin{pmatrix} u \\ V_1u \\ V_2u \end{pmatrix} = A_j \begin{pmatrix} u \\ V_1u \\ V_2u \end{pmatrix} \quad (j = 0, 1, 2)$$

where A_0, A_1, A_2 are matrices of rank three defined as follows.

```
A0={{s0,0,0},{0,3+s0,0},{0,0,4+s0}};
A1={{0,1,0},{1/2099520*(320*(1+1728*h1-4*s0)*(3+864*h1+s0)*x1^6-
    120*(47+58752*h1-280*s0)*(3+864*h1+s0)*x1^4*x2+36*(2115+
    2967840*h1+679311360*h1^2-15870*s0-2515104*h1*s0-
    6125*s0^2)*x1^2*x2^2-3888*(15+30240*h1+7464960*h1^2-175*s0+
    28512*h1*s0)*x2^3+45*(3+864*h1-35*s0)*x1*x3),-1/810*x1*((55+
    42768*h1+40*s0)*x1^2-3*(255+83376*h1+175*s0)*x2),-1/6*h1*(x1^2-
    6*x2)},{1/656100*(-25280*(1+1728*h1-4*s0)*(3+864*h1+s0)*x1^7
    +120*(9879+16458336*h1+3920596992*h1^2-42907*s0-16232832*h1*s0-
    17560*s0^2)*x1^5*x2-36*(123660+210094560*h1+50250378240*h1^2-
    721860*s0-210720096*h1*s0-308135*s0^2)*x1^3*x2^2+1944*(345+3546720*h1+
    992839680*h1^2-14815*s0+6258816*h1*s0-5775*s0^2)*x1*x2^3-45*(147+
    42336*h1-1625*s0)*x1^2*x3-3645*(5+1440*h1+44*s0)*x2*x3),(4*(316*(-15+
    25704*h1+10*s0)*x1^4-3*(-16645+15303168*h1+8125*s0)*x1^2*x2-2430*(56+
    1440*h1-11*s0)*x2^2))/2025,(x1*(2*(-85+42768*h1-20*s0)*x1^2-3*(-715
    +166752*h1-175*s0)*x2))/1620}};
A2={{0,0,1},{1/656100*(-25280*(1+1728*h1-4*s0)*(3+864*h1+s0)*x1^7+120*(9879+
```

```
16458336*h1+3920596992*h1^2-75307*s0-16232832*h1*s0-17560*s0^2)*x1^5*x2-
36*(123660+210094560*h1+50250378240*h1^2-1275765*s0-210720096*h1*s0-
308135*s0^2)*x1^3*x2^2+1944*(345+3546720*h1+992839680*h1^2-3530*s0+
6258816*h1*s0-5775*s0^2)*x1*x2^3-45*(147+42336*h1-3785*s0)*x1^2*x3-
6075*(3+864*h1-35*s0)*x2*x3),4*(632*(15+12852*h1+5*s0)*x1^4-3*(24375+
15303168*h1+8125*s0)*x1^2*x2-2430*(-33+1440*h1-11*s0)*x2^2)/2025,x1*(2*
(5+42768*h1-20*s0)*x1^2-3*(15+166752*h1-175*s0)*x2)/1620},{1/820125*(4*
(1997120*(1+1728*h1-4*s0)*(3+864*h1+s0)*x1^8-120*(680901+1246968864*h1+
302650380288*h1^2-3612193*s0-1421635968*h1*s0-1027000*s0^2)*x1^6*x2+
36*(6514065+14746523040*h1+3706696028160*h1^2-67397805*s0-17324851104*h1*
s0-16957205*s0^2)*x1^4*x2^2-972*(-235065+392096160*h1+132420925440*h1^2
-3712385*s0+1286813088*h1*s0-1072500*s0^2)*x1^2*x2^3+1574640*(5+1440*h1
-11*s0)*(-5+8640*h1+33*s0)*x2^4+45*(4503+1296864*h1-144155*s0)*x1^3*x3+
3645*(740+213120*h1+5891*s0)*x1*x2*x3)),-16*(49928*(-25+60048*h1)*x1^5-
192*(-44815+84795282*h1)*x1^3*x2-1620*(4469+1982880*h1)*x1*x2^2
+180225*x3)/10125,-8*(632*(-10+6426*h1-5*s0)*x1^4-3*(-16250+7651584*h1-
8125*s0)*x1^2*x2-2430*(22+720*h1+11*s0)*x2^2)/2025}};
```

Remark 6.2 *A_1, A_2, A_3 contain parameters s_0, h_1. The determination of A_1, A_2, A_3 was accomplished by Masayuki Noro (Kobe Univ.).*

The case $s_0 = \frac{1}{6},\ h_1 = -\frac{19}{5184}$

In this case the monodromy group of the system of differential equations becomes G_{2160}. This case is treated in [6].

The case $s_0 = -3,\ h_1 = 0$

In this case there is a quotient of the system above. In fact,

$$\begin{cases} V_1 u & = & \frac{1}{162} x_1(13x_1^2 - 162x_2)u \\ V_j \begin{pmatrix} u \\ V_2 u \end{pmatrix} & = & B_{j+1} \begin{pmatrix} u \\ V_2 u \end{pmatrix} \quad (j = 0, 1, 2) \end{cases}$$

is a quotient of the system $V_j \vec{u} = A_j \vec{u}$ $(j = 0, 1, 2)$ defined above, where B_j $(j = 0, 1, 2)$ are matrices of rank two defined below:

```
B0={{-3,0},{0,1}};
B1={{1/162*x1*(13*x1^2-162*x2),0},
   {(-98592*x1^7+1926304*x1^5*x2-10970316*x1^3*x2^2+17754552*x1*x2^3
   -15066*x1^2*x3+30861*x2*x3)/43740,(-1350*x1^3+15390*x1*x2)/43740}};
B2={{0,1},{-1/164025*(4*(-6490640*x1^8+180214176*x1^6*x2-999084132*x1^4*x2^2+
    712058040*x1^2*x2^3+1244595456*x2^4-2995542*x1^3*x3+665577*x1*x2*x3)),
    -4*(511920*x1^4-3948750*x1^2*x2+4330260*x2^2)/164025}};
```

Bibliography

1. A.G.Aleksandrov. Milnor numbers of nonisolated Saito singularities. Funct. Anal. Appl., 21 (1987), 1-9.

2. A. G. Aleksandrov, Nonisolated hypersurfaces singularities. Advances in Soviet Math., 1 (1990), 211-246.

3. A. G. Aleksandrov. Moduli of logarithmic connections along free divisor. Contemp. Math., 314 (2002), 2-23.

4. A. G. Aleksandrov, J. Sekiguchi, Free deformations of hypersurface singularities, To appear in Kokyuroku.

5. D. Bessis, J. Michel. Explicit presentations for exceptional braid groups, Experimental Math., 13 (2004), 257-266.

6. Y. Haraoka, M. Kato. Systems of partial differential equations with finite monodromy groups, preprint.

7. F. Klein. Vorlesungen über das Ikosaeder, Birkhäuser Verlag, Basel-Boston-Berlin, 1993.

8. P. Orlik, H. Terao. Arrangements of Hyperplanes, Grundlehren der mathematisches Wissenschaften 300, Berlin: Springer-Verlag, 1992.

9. K. Saito. On the uniformization of complements of discriminant loci. RIMS Kokyuroku 287 (1977), 117-137.

10. K. Saito. Theory of logarithmic differential forms and logarithmic vector fields. J. Faculty of Sciences, Univ. Tokyo 27 (1980), 265-291.

11. K. Saito. On a linear structure of the quotient variety by a finite reflection group. Publ. Res. Inst. Math. Sci. Kyoto Univ., 29 (1993), 535-579.

12. K. Saito. Uniformization of orbifold of a finite reflection group, Frobenius Manifold, Quantam Cohomology and Singularities. A Publication of the Max-Planck-Institute, Mathematics, Bonn, 265-320.

13. J. Sekiguchi. A classification of weighted homogeneous Saito free divisors, J. Math. Soc. Japan, 61 (2009), 1071-1095.

14. J. Sekiguchi. Three dimensional Saito free divisors and deformations of singular curves, J. Siberian Federal Univ., Mathematicas and Physics, 1 (2008), 33-41.

15. J. Sekiguchi. Systems of uniformization equations and hyperelliptic integrals, preprint.

16. G. C. Shephard, A. J. Todd. Finite reflection groups, Canad. J. Math., 6 (1954), 274-304.

17. T. Yano, J. Sekiguchi. The microlocal structure of weighted homogeneous polynomials associated with Coxeter systems. I, II, Tokyo J. Math. 2 (1979), 193-219, Tokyo J. Math. 4 (1981), 1-34.

ГРУППЫ ОТРАЖЕНИЙ РАНГА ТРИ И СИСТЕМЫ УРАВНЕНИЙ ДИФОРМАЛИЗАЦИИ

Джиро Секигучи

Токийский университет сельского хозяйства и технологии,
Коганей, Токио, 184-8588, Япония, e-mail: sekiguti@cc.tuat.ac.jp

Аннотация. В работе приведены результаты, описывающие дискриминант неприводимых комплексных групп отражения ранга три.

Ключевые слова: неприводимые уравнения, дискриминант комплексных групп отражения.

УДК 519.6

THE MULTIPOLE METHOD FOR CERTAIN ELLIPTIC EQUATION WITH DISCONTINUOUS COEFFICIENT

S.L. Skorokhodov, V.I. Vlasov

Dorodnicyn Computing Centre of the Russian Academy of Sciences,

Vavilov str., 40, Moscow, 119991, Russia, e-mail: skor@ccas.ru,vlasov@ccas.ru

Abstract. A new analytic–numerical method has been developed for solving BVPs in 3D domains with cones of arbitrary base for certain elliptic equation with piecewise constant coefficient. The solution is obtained by the use of special basic functions — the Multipoles, which are costructed in an explicit form. The method supplies high accurate evaluation of the solution, its derivatives, singularity exponents and intensity factors near the geometrical singularities — edges and the corner vertex.

Keywords: boundary value problem, domains with cones, multipole method, singularity exponents, intensity factors.

1 Introduction

We consider boundary value problems (BVPs) for certain elliptic equation with piecewise constant coefficient in domains with cones of arbitrary base (particularly, with polyhedral corners), when the surface of discontinuity of the coefficient (interface surface) is a conical one passing through the vertex of the initial cone. An equivalent statement, which is called a transmission problem, consists in solving the Laplace equation with so called transmission conditions on the interface surface [1].

Solutions of such BVPs have singularities at vertices of the cones [2]-[8]. Development of effective methods for solving these BVPs, including high accuracy computation of the singularity exponents, became a challenging issue [1], [9]-[13].

In this work we present a new effective analytic–numerical method for high–accuracy computation of these singularities at cones of arbitrary base (in particular, for polyhedral corners), when conical interface surface also has an arbitrary base. This method represents a generalization of the Multipole method, previously developed in [14]-[16] for solving a certain class of 2D and 3D elliptic BVPs in domains of complex shape with geometric singularities of different kinds. For the case of the Laplace equation, the Multipole method in domains with cones has been developed in [17]-[19].

The principle underlying our method consists in using a system of basic functions that conform to the structure of the solution near the conical surfaces of the boundary and interface. We call these functions Multipoles due to their similarity to ordinary multipoles, known in the theory of potential [20]. Such systems possess good approximation properties. Most important is the fact that these basic functions can be expressed in explicit analytic form in terms of special functions.

By virtue of these features the method proves most effective for precise computation of exponents at the cone singularity.

The work is supported by RFBR (projects 07-01-00295, 07-01-00503) and by Program №3 of Fundamental Research of the Mathematical Sciences Department of RAS.

2 Statement of the problem

2.1 Domains $\mathcal{K}$ and Ω

Let (r, θ, φ) be spherical co-ordinates of a point x in space $\mathbb{R}^3$. Denote by

$$\mathbb{S}^2 := \{r = 1,\, \theta \in [0, \pi],\, \varphi \in [0, 2\pi)\}$$

the unit sphere and by $\mathbb{B}^3$ the unit ball in $\mathbb{R}^3$. Points $\theta = 0$ and $\theta = \pi$ on $\mathbb{S}^2$ are called the North Pole $\mathcal{P}_N$ and South Pole $\mathcal{P}_S$, respectively.

Consider two disjoint Lipschitz piecewise smooth contours $\mathcal{L}$ and $\mathcal{L}_{in}$ on the sphere $\mathbb{S}^2$, each divides $\mathbb{S}^2$ into two domains, one of which contains $\mathcal{P}_S$ and another $\mathcal{P}_N$. The domain containing $\mathcal{P}_N$ and bounded by $\mathcal{L}$ (by $\mathcal{L}_{in}$) is denoted by $\mathcal{S}$ (by $\mathcal{S}_{in}$). Assume that $\mathcal{L}_{in} \subset \mathcal{S}$ and denote $\mathcal{S}_{ex} = \mathcal{S} \setminus \overline{\mathcal{S}}_{in}$; observe that $\partial\mathcal{S}_{ex} = \mathcal{L} \cup \mathcal{L}_{in}$.

The domain $\mathcal{K} \subset \mathbb{R}^3$ defined by the formula $\mathcal{K} := \{r \in (0, \infty),\, (\theta,\, \varphi) \in \mathcal{S}\}$ is an (infinite) cone with base $\mathcal{S}$, its boundary being the conical surface $\partial\mathcal{K} = \{r \in (0, \infty),\, (\theta,\, \varphi) \in \mathcal{L}\}$. The domains $\mathcal{K}_{in}, \mathcal{K}_{ex}$ and their boundaries $\partial\mathcal{K}_{in}$, $\partial\mathcal{K}_{ex}$ are defined in a similar way, with the vertex $\{0\}$ shared by both cones, $\overline{\mathcal{K}} = \overline{\mathcal{K}}_{in} \cup \overline{\mathcal{K}}_{ex}$, conical surface $\partial\mathcal{K}_{in}$ contained in $\mathcal{K} \cup \{0\}$, and $\partial\mathcal{K}_{ex} = \partial\mathcal{K} \cup \partial\mathcal{K}_{in}$.

Consider an important instance of cone $\mathcal{K}$ when it presents a trihedral corner with its three faces being plane angles with common vertex $\{0\}$ and with values of the angles being equal to $\pi\alpha$, where $\alpha \in (0, 2/3]$. Denote by $\mathcal{K}^\alpha$ this trihedral corner, by $\mathcal{S}^\alpha$ its base, and by $\mathcal{L}^\alpha$ the contour of this base. In this instance, the equation of contour $\mathcal{L}^\alpha$ can be written in the form

$$\mathcal{L}^\alpha = \{(\theta, \varphi) : \theta = \theta(\varphi),\, \varphi \in [0, 2\pi)\},\;\; \theta(\varphi) = \begin{cases} \mathrm{T}(\varphi + \frac{2\pi}{3});\;\; \varphi \in [0,\, \frac{2\pi}{3}], \\ \mathrm{T}(\varphi);\quad \varphi \in [\frac{2\pi}{3},\, \frac{4\pi}{3}], \\ \mathrm{T}(\varphi - \frac{2\pi}{3});\;\; \varphi \in [\frac{4\pi}{3},\, 2\pi], \end{cases} \tag{2.1}$$

with function $\mathrm{T}\,(\varphi)$ given by the formula

$$\mathrm{T}\,(\varphi) \;=\; \arccos \left[\cos\varphi / \sqrt{c + \cos^2\varphi} \right] \tag{2.2}$$

that involves parameter $c = (1 - \cos\pi\alpha)\,\big(2 + 4\cos\pi\alpha\big)^{-1}$. It worth to be mentioned that value $\pi\beta$ of dihedral angle between faces of $\mathcal{K}^\alpha$ are related to the quantity $\pi\alpha$ by the formula $\cos\pi\alpha = \cos\pi\beta\,/(1 - \cos\pi\beta)$.

The transmission BVP is being solved in a domain $\Omega \subset \mathcal{K}$ homeomorphic to $\mathbb{B}^3$ with Lipschitz piecewise smooth boundary $\partial\Omega$. By definition, boundary $\partial\Omega$ consists of the two disjoint parts: $\partial\Omega = \gamma \cup \Gamma$, where γ is a closure of a simply-connected domain on the conical surface $\partial\mathcal{K}$ with its vertex $\{0\}$ being an interior point of γ, and $\Gamma \subset \mathcal{K}$ is a simply-connected domain on a certain piecewise smooth surface. Note that $\mathcal{K}$ is an extension of Ω through Γ. Assume that $\partial\mathcal{K}_{in}$ divides Ω into two subdomains Ω_{in} and Ω_{ex}. Define $\gamma_{in} = \partial\mathcal{K}_{in} \cap \overline{\Omega}$ and observe that $\gamma_{in} = \partial\Omega_{in} \cap \partial\Omega_{ex}$. Note that γ_{in} is the interface surface within domain Ω where the transmission conditions are to be set.

Let the surface Γ be divided by a Lipschitz piecewise smooth curve or contour into two domains: $\mathcal{D}$ and $\mathcal{N}$; the latter correspond to the boundary conditions (the Dirichlet or Neumann type) to be set on the corresponding parts of Γ.

2.2 The formulation of the transmission BVP with mixed Dirichlet — Neumann boundary conditions

For a function ψ defined on Ω, denote by ψ_{in} (by ψ_{ex}) its restriction to Ω_{in} (to Ω_{ex}). Consider the following transmission BVP for the Laplace equation in the domain Ω:

$$\Delta\,\psi_{in} = 0 \quad \text{in } \ \Omega_{in}\,, \qquad \Delta\,\psi_{ex} = 0 \quad \text{in } \ \Omega_{ex}\,, \tag{2.3}$$

with the transmission conditions on the interface surface

$$\psi_{in}\big|_{\gamma_{in}} = \psi_{ex}\big|_{\gamma_{in}}, \qquad \varkappa_{in}\,\partial_\nu\,\psi_{in}\big|_{\gamma_{in}} = \varkappa_{ex}\,\partial_\nu\,\psi_{ex}\big|_{\gamma_{in}}\,, \tag{2.4}$$

where $\varkappa_{in}$ and $\varkappa_{ex}$ are prescribed positive constants, ∂_ν being a normal derivative, and with mixed Dirichlet — Neumann type conditions

$$\psi\,\big|_{\gamma} = 0\,, \qquad \psi\,\big|_{\mathcal{D}} = h_{\mathcal{D}}, \qquad \partial_\nu\psi\,\big|_{\mathcal{N}} = h_{\mathcal{N}} \tag{2.5}$$

on the boundary $\partial\Omega = \gamma\cup\Gamma$.

We shall use the notation $h(x)$ defined by equalities

$$h(x) \,=\, h_{\mathcal{D}}(x), \quad x\,\in\,\mathcal{D}; \qquad h(x) \,=\, h_{\mathcal{N}}(x), \quad x\,\in\,\mathcal{N}, \tag{2.6}$$

and notation $\varkappa$ defined by the formula

$$\varkappa \,=\, \varkappa_{in}, \quad x\,\in\,\Omega_{in}; \qquad \varkappa \,=\, \varkappa_{ex}, \quad x\,\in\,\Omega_{ex}. \tag{2.7}$$

Transmission problem (2.3)–(2.5) can be rewritten in a generalized statement [6]-[8], [21]-[24]. In order to do it, Sobolev spaces are introduced, following [23]-[26].

Denote by $\overset{\circ}{W}{}^1_2(\Omega,\,\gamma)$ a subspace of $W^1_2(\Omega)$ consisting of functions having zero trace on γ. Similarly, define the space $\overset{\circ}{W}{}^1_2(\Omega,\gamma\cup\mathcal{D})$ as a subspace of $W^1_2(\Omega)$ consisting of functions with zero trace on $\gamma\cup\mathcal{D}$.

Let A be a subdomain of boundary $\partial\Omega$, and let a be a subdomain of A. Denote by $\overset{\circ}{W}{}^{1/2}_2(A,a)$ a subspace of the Sobolev — Slobodetskii space $W^{1/2}_2(A)$ consisting of functions vanishing a.e. on a. Only the particular cases of the latter spaces $\overset{\circ}{W}{}^{1/2}_2(\gamma\cup\mathcal{D},\gamma)$ and $\overset{\circ}{W}{}^{1/2}_2(\partial\Omega,\,\gamma\cup\mathcal{D})$ are to be employed below. The so called negative space $\overset{\circ}{W}{}^{-1/2}_2(\partial\Omega,\gamma\cup\mathcal{D})$ is defined as a conjugate space to $\overset{\circ}{W}{}^{1/2}_2(\partial\Omega,\gamma\cup\mathcal{D})$.

The boundary data $h_{\mathcal{D}}$ and $h_{\mathcal{N}}$ in conditions (2.5) are required to belong to the spaces

$$h_{\mathcal{D}}\,\in\,\overset{\circ}{W}{}^{1/2}_2(\gamma\cup\mathcal{D},\gamma), \qquad h_{\mathcal{N}}\,\in\,\overset{\circ}{W}{}^{-1/2}_2(\partial\Omega,\gamma\cup\mathcal{D}). \tag{2.8}$$

A generalized solution of BVP (2.3)–(2.5) is understood to be a function $\psi\,\in\,\overset{\circ}{W}{}^1_2(\Omega,\gamma)$ satisfying boundary condition $\psi\big|_{\mathcal{D}} = h_{\mathcal{D}}$ and the integral identity

$$\int_\Omega \varkappa\,(\nabla\psi,\,\nabla\eta)\,dx \,=\, \int_{\mathcal{N}} h_{\mathcal{N}}\,\eta\,ds$$

for all test-functions $\eta\in\overset{\circ}{W}{}^1_2(\Omega,\gamma\cup\mathcal{D})$, where the notation $(\,.\,,\,.\,)$ stands for the inner product in Euclidean space $\mathbb{R}^3$, and $\varkappa$ is defined by (2.7).

Solvability of the formulated BVP is guaranteed by the following

Theorem 1. *For any* $h_{\mathcal{D}}$ *and* $h_{\mathcal{N}}$ *satisfying (2.8) there exists a unique generalized solution* $\psi \in \overset{\circ}{W}{}^1_2(\Omega, \gamma)$ *of the problem (2.3)–(2.5).*

It is clear that Theorem 1 admits a standard proof which reduces to the Riesz representation theorem and follows well-known patterns (see e.g. [24]). Outside the boundary's singularities, regularity of the generalized solution of (2.3)–(2.5) is covered by the standard theory of elliptic BVPs [3], [6], [8], [21], [23]. Namely, the generalized solution is infinitely differentiable at any interior point $x \in \Omega \setminus \gamma_{in}$ as well as at any interior point of γ. At γ_{in}, the generalized solution is differentiable one-sidedly, i.e. on either side of γ_{in}, as many times as allows the smoothness of γ_{in}. Omitting the details, we just mention that regularity of the generalized solution at boundary points $x \in \mathcal{D}$ and $x \in \mathcal{N}$ depends on the smoothness of boundary surface Γ and boundary data $h_{\mathcal{D}}$, $h_{\mathcal{N}}$.

3 Construction of the system of basic functions (the Multipoles)

3.1 Reduction to a spectral problem for the Beltrami — Laplace operator with transmission conditions

Our goal consists in constructing a system of functions Ψ_k (the Multipoles) that possess good approximation properties and conform to the structure of the solution near the conical surfaces, which contain singularities (the vertex and edges). The basic functions are defined on the whole cone domain $\mathcal{K}$; their restrictions to $\mathcal{K}_{in}$ and $\mathcal{K}_{ex}$ are denoted by $\Psi_{k,in}$ and $\Psi_{k,ex}$, respectively. The desired properties of the basic functions require the following conditions to be met: 1) functions Ψ_k identically satisfy the Laplace equation in $\mathcal{K}$ with transmission conditions (2.4) on $\partial\mathcal{K}_{in}$; 2) they identically meet the homogeneous Dirichlet condition $\Psi_k = 0$ on $\partial\mathcal{K}$; 3) they constitute an orthogonal basis in $L_2(\mathcal{S})$.

The Multipoles are represented in the form

$$\Psi_k(r,\, \theta,\, \varphi) \,=\, r^{\mu}\, U(\mu;\, \theta,\, \varphi), \qquad \mu \,=\, \mu(k)\,; \qquad k \,=\, 1,\, 2,\, \ldots\,; \tag{3.1}$$

restrictions of $U(\mu; \theta, \varphi)$ to $\mathcal{S}_{in}$ and to $\mathcal{S}_{ex}$ are denoted by U_{in} and by U_{ex}, respectively.

Thus $U(\mu(k); \theta, \varphi) = U_k$ are eigenfunctions with eigenvalues $\mu(k)$ for the Laplace — Beltrami operator in domain $\mathcal{S}$ on the unit sphere

$$\frac{1}{\sin\theta}\,\frac{\partial}{\partial\,\theta}\left(\sin\theta\,\frac{\partial U}{\partial\,\theta}\right) + \frac{1}{\sin^2\theta}\,\frac{\partial^2 U}{\partial^2\,\varphi} \,+\, \mu\,(\mu\,+\,1)\,U \,=\, 0 \qquad \text{in } \mathcal{S}\setminus\mathcal{L}_{in}\,, \tag{3.2}$$

with the transmission conditions on interface line $\mathcal{L}_{in}$, induced by (2.4), and with homogeneous Dirichlet condition on $\mathcal{L}$:

$$U_{in}\big|_{\mathcal{L}_{in}} = U_{ex}\big|_{\mathcal{L}_{in}}, \qquad \varkappa_{in}\,\partial_\nu\, U_{in}\big|_{\mathcal{L}_{in}} = \varkappa_{ex}\,\partial_\nu\, U_{ex}\big|_{\mathcal{L}_{in}}, \qquad U\big|_{\mathcal{L}} = 0\,. \tag{3.3}$$

Denote by $\overset{\circ}{W}{}^1_2(\mathcal{S})$ a subspace of $W^1_2(\mathcal{S})$ consisting of functions having zero trace on $\mathcal{L}$. A generalized solution of BVP (3.2), (3.3) is understood to be a function $U \in \overset{\circ}{W}{}^1_2(\mathcal{S})$ satisfying the integral identity

$$\int_{\mathcal{S}} \varkappa\,(\,\nabla_{\mathcal{S}}\, U,\ \nabla_{\mathcal{S}}\, V\,)\, ds \,=\, \mu\,(\mu+1)\int_{\mathcal{S}} U\, V\, ds \qquad \forall\; V \,\in\, \overset{\circ}{W}{}^1_2(\mathcal{S}), \tag{3.4}$$

where $\nabla_{\mathcal{S}}$ stands for a tangential component to $\mathcal{S}$ of the gradient ∇. Note that an inner product

$$[U,\, V]_{\mathcal{S}} \,=\, \int_{\mathcal{S}} \varkappa\,(\nabla_{\mathcal{S}}\, U,\, \nabla_{\mathcal{S}}\, V\,)\, ds$$

induces an equivalent norm on $\overset{\circ}{W}{}^1_2(\mathcal{S})$.

Theorem 2. *For a spectral problem (3.4), there exists a denumerable set of generalized solutions* $U = U_k \in \overset{\circ}{W}{}^1_2(\mathcal{S})$, $\mu = \mu(k)$, $k = 1, 2, \ldots$. *The eigenvalues* $\mu(k)$ *have no finite limit points, and* $\mu(k) \to \infty$ *as* $k \to \infty$. *To each eigenvalue* $\mu(k)$ *there corresponds at most a finite number of generalized eigenfunctions* $U_k \in \overset{\circ}{W}{}^1_2(\mathcal{S})$. *The eigenfunctions* $\{U_k\}$ *form a basis in* $L_2(\mathcal{S})$ *and* $W^1_2(\mathcal{S})$, *which is orthonormal in* $L_2(\mathcal{S})$ *and orthogonal with respect to the inner product* $[\,.\,,\,.\,]_{\mathcal{S}}$.

It is clear that Theorem 2 admits a standard proof following the pattern of [21].

Remark 1. *In accordance with Theorem 2, all eigenvalues* $\mu(k)$, $k = 1,\, 2,\, \ldots$ *can be enumerated in order of their nondecresing; each multiple eigenvalue should be counted accordind to its multiplicity. Such renumbering establishes a one-to-one correspondence between eigenvalues* $\mu(k)$ *and eigenfunctions* U_k.

3.2 Solution of the spectral problem

In what follows we restrict ourselves to the case of contours $\mathcal{L}$, $\mathcal{L}_{in}$ being star–like on $\mathbb{S}^2$ with respect to North Pole, when $\mathcal{L}$ can be represented in the form

$$\mathcal{L} \,=\, \{(\theta,\, \varphi)\,:\, \theta \,=\, \theta\,(\varphi),\;\; \theta\,(\varphi) \,\in\, C(-\infty,\, +\infty),\;\; \theta\,(\varphi) \,=\, \theta\,(\varphi \,+\, 2\,\pi)\,\}$$

and $\mathcal{L}_{in}$ can be represented in a similar form.

The eigenfunctions of the problem (3.2), (3.3) are constructed using two systems of complex–valued functions: $\{u^m(\mu;\, \theta,\, \varphi)\,\}_{m=0}^{\infty}$ and $\{\, v^m(\mu;\, \theta,\, \varphi)\,\}_{m=0}^{\infty}$ defined by the formulas:

$$u^{\,m}(\mu;\, \theta,\, \varphi) = \mathrm{P}^{\,m}_{\mu}(\cos\, \theta)\, e^{\,i\,m\,\varphi}, \qquad v^{\,m}(\mu;\, \theta,\, \varphi) = \mathrm{P}^{\,m}_{\mu}(-\cos\, \theta)\, e^{\,i\,m\,\varphi}, \tag{3.5}$$

where $\mathrm{P}^m_\mu(t)$ are associated Legendre functions on the cut [27]. For short, in complicated expressions we reduce the relations (3.5) to $u^m\,(\mu)$, $v^m\,(\mu)$.

Note that if $\mathcal{K}$ is a circular cone, i.e. $\mathcal{L}$ is a circumference $\{\theta \,=\, \theta_0 \,=\, \mathrm{const}\}$, then $\mathrm{Re}\; u^m(\mu;\theta,\varphi)$ and $\mathrm{Im}\; u^m(\mu;\theta,\varphi)$ are eigenfunctions of the problem (3.2), (3.3) with $\mu = \mu^m_n$ being the root of number $n\,(n \,=\, 1,\, 2,\, \ldots)$ of the equation $\mathrm{P}^m_\mu(\cos\theta_0) \,=\, 0$. Taking this fact into account we rename and renumber eigenvalues $\mu(k)$ as μ^m_n and eigenfunctions $U\big(\mu(k);\, \theta,\varphi\big)$ as $U^{m\,+}_n(\theta,\varphi)$ and $U^{m\,-}_n(\theta,\varphi)$.

Denoting restrictions of $U^{m\,\pm}_n(\theta,\, \varphi)$ to $\mathcal{S}_{in}$ and $\mathcal{S}_{ex}$ by $U^{m\,\pm}_{n,\,in}(\theta,\, \varphi)$ and $U^{m\,\pm}_{n,\,ex}(\theta,\, \varphi)$, respectively, let represent the desired eigenfunctions in the form of expansions in terms of functions (3.5):

$$U^{m\,\pm}_{n,\,in} \,=\, \mathrm{Re}\, \sum\nolimits_{l=0}^{\infty} A^{m,\,l\,\pm}_{n}\, u^{\,m\,+\,l}(\mu), \qquad A^{m,\,0\,+}_{n} \,=\, 1, \qquad A^{m,\,0\,-}_{n} \,=\, i, \tag{3.6}$$

$$U^{m\,\pm}_{n,\,ex} \,=\, \mathrm{Re}\, \sum\nolimits_{l=0}^{\infty} \Big\{\, B^{m,\,l\,\pm}_{n}\, u^{\,m\,+\,l}(\mu) \,+\, C^{m,\,l\,\pm}_{n}\, v^{\,m\,+\,l}(\mu)\,\Big\}, \quad \mu \,=\, \mu^m_n. \tag{3.7}$$

Observe that functions (3.6), (3.7) with any coefficients identically satisfy the equation (3.2). Unknown eigenvalues μ_n^m and coefficients $A_n^{m,l\,\pm}$, $B_n^{m,l\,\pm}$, $C_n^{m,l\,\pm}$ in representations (3.6), (3.7) should be found from relations (3.3), which unite the transmission conditions on interface line $\mathcal{L}_{in}$ and boundary condition on outer contour $\mathcal{L}$.

We shall make it in the following way. Functions $U_n^{m\,\pm}(\theta,\,\varphi)$ are sought as a limit

$$U_n^{m\,\pm}(\theta,\,\varphi) \;=\; \lim_{M\to\infty} U_n^{m\,\pm}(M;\,\theta,\,\varphi)$$

of consequent approximations $U_n^{m\,\pm}(M;\,\theta,\,\varphi)$ written in the form of finite sums (3.6), (3.7) with coefficients depending on the length M of approximation, i.e.

$$U_{n,\,in}^{m\,\pm}(M;\theta,\varphi) = \mathrm{Re}\sum_{l=0}^{M} A_n^{m,\,l\,\pm}(M)u^{m+l}, \quad A_n^{m,\,0+}(M) = 1,\;\; A_n^{m,\,0-}(M) = i, \tag{3.8}$$

$$U_{n,\,ex}^{m\,\pm}(M;\,\theta,\,\varphi) \;= \mathrm{Re}\,\sum\nolimits_{l=0}^{M}\Big\{B_n^{m,\,l\,\pm}(M)\,u^{\,m+l} \;+\; C_n^{m,\,l\,\pm}(M)\,v^{\,m+l}\Big\}\,. \tag{3.9}$$

Coefficients $A_n^{m,\,l\,\pm}(M)$, $B_n^{m,\,l\,\pm}(M)$, $C_n^{m,\,l\,\pm}(M)$ and approximate eigenvalues $\mu_n^{m\,\pm}(M)$ are determined by substituting $U_n^{m\,\pm}(M)$ into the transmission and boundary conditions (3.3) and by projecting the result onto the system of trigonometric functions $\exp(iq\varphi)$:

$$\Big(U_{n,\,ex}^{m\pm}(M),\;\exp(iq\varphi)\Big)_{\mathcal{L}} = 0, \qquad \Big(U_{n,\,ex}^{m\pm}(M) - U_{n,\,in}^{m\,\pm}(M),\;\exp(iq\varphi)\Big)_{\mathcal{L}_{in}} = 0, \tag{3.10}$$

$$\Big(\,\varkappa_{ex}\,\partial U_{n,\,ex}^{\,m\,\pm}(M)\,/\,\partial\nu \;-\; \varkappa_{in}\,\partial U_{n,\,in}^{\,m\,\pm}(M)\,/\,\partial\nu,\;\exp\,(i\,q\,\varphi)\,\Big)_{\mathcal{L}_{in}} \;=\; 0, \tag{3.11}$$

where $q = m, \ldots, m+M$, and $(f_1,\,f_2)_{\mathcal{L}}$ or $(f_1,\,f_2)_{\mathcal{L}_{in}}$ is the inner product in $L_2(\mathcal{L})$ or in $L_2(\mathcal{L}_{in})$. Substituting representations (3.8), (3.9) into relations (3.10), (3.11) we obtain a system of linear equations with respect to coefficients $A_n^{m,\,l\,\pm}(M)$, $B_n^{m,\,l\,\pm}(M)$, $C_n^{m,\,l\,\pm}(M)$:

$$\mathcal{D}^{\,m}(\mu)\,\mathcal{Z} \;=\; 0, \tag{3.12}$$

where

$$\mathcal{Z} = \Big[A_n^{m,0\pm}(M), B_n^{m,0\pm}(M), C_n^{m,0\pm}(M), ..., A_n^{m,M\pm}(M), B_n^{m,M\pm}(M), C_n^{m,M\pm}(M)\Big]^T$$

is a vector of the coefficients. Elements of matrix $\mathcal{D}^m(\mu)$ of system (3.12) are expressed as integrals over contours $\mathcal{L}$ or $\mathcal{L}_{in}$ of products of functions (3.5) or there normal derivatives on $\mathcal{L}_{in}$; so, these elements depend only on number m and parameter μ.

In order to find a nontrivial solutions of homogeneous system (3.12), we equate the determinant of its matrix to zero, and in the issue we obtain the relation $\det \mathcal{D}^{\,m}(\mu) \;=\; 0$, which should be considered as a transcidental equation with respect to μ. So, eigenvalue $\mu_n^m(M)$ is a root of number n ($n \;=\; 1,\, 2,\, \ldots$) of this equation.

The performed numerical experiments showed that the approximate eigenvalues and eigenfunctions converge to the exact ones. Namely, there hold the relations: 1) for any compact $E \subset \mathcal{S}$ it holds

$$\lim_{M\to\infty}\left[\;\max_{(\theta,\,\varphi)\in E}\Big|U_n^{\,m\,\pm}(M;\,\theta,\,\varphi) \;-\; U_n^{\,m\,\pm}(\theta,\,\varphi)\Big|\;\right] = 0;$$

2) for all coefficients in (3.8), (3.9) it holds

$$A_n^{m,\,l\pm}(M) \to A_n^{m,\,l\pm}, \quad B_n^{m,\,l\pm}(M) \to B_n^{m,\,l\pm}, \quad C_n^{m,\,l\pm}(M) \to C_n^{m,\,l\pm} \text{ as } M \to \infty;$$

3) for all eigenvalues it holds $\mu_n^m(M) \to \mu_n^m$ as $M \to \infty$.

3.3 Computation of integrals of frequently oscillating functions

One of important computational problems arising in the described algorithm is calculation of elements of matrix $\mathcal{D}^m(\mu)$ of system (3.12); those elements are expressed in the form of integrals over contours $\mathfrak{L}$ or $\mathfrak{L}_{in}$ of the following type:

$$\int \mathrm{P}_\mu^a\big(\cos\theta(\varphi)\big)\,\exp(\,ib\varphi\,)\,d\varphi\,, \tag{3.13}$$

where $\theta(\varphi)$ is an equation of the contour; a and b are natural numbers, possibly very large. So, (3.13) are integrals with frequently oscillating integrand; effective computation of those integrals is a well–known challenging problem. A special analytic–numerical method has been developed for computation of such integrals. This method represents integrals (3.13) as exponentially convergent series involving integrals $\int_0^{\pi/2}\cos^\alpha t\cos(\beta t)\,dt$ and related ones, which we have computed explicitly via special functions, for whose computation high effective methods have been developed [28]. Particularly,

$$\int_0^{\pi/2}\cos^\alpha t\cos(\beta t)\,dt \;=\pi\,(1\,+\,\alpha)\,2^{-1-\alpha}\left[\mathrm{B}\left(1+\frac{\alpha+\beta}{2},\,1+\frac{\alpha-\beta}{2}\right)\right]^{-1},$$

where $\mathrm{B}(x,\,y)$ is Beta–function [27].

3.4 Numerical results

Note, that input data for the spectral transmission BVP (3.2), (3.3) consist, at first, of geometric data, determined by outer contour $\mathfrak{L}$ and interface line $\mathfrak{L}_{in}$, and, at second, of mechanical quantity $\kappa := \varkappa_{in}/\varkappa_{ex}$.

The method of solving this problem described in Sect. 3.2 has been realized for two types of geometric data. For type I contour $\mathfrak{L}$ is $\mathfrak{L}^\alpha$ turned to the angle δ $\mathfrak{L} = \{(\theta,\,\varphi) : (\theta,\,\varphi-\delta) \in \mathfrak{L}^\alpha\}$, and $\mathfrak{L}_{in} = \mathfrak{L}^{\alpha_{in}},\ \alpha_{in} > \alpha$. Remind, that contour $\mathfrak{L}^\alpha$ is defined by (2.1), (2.2).

For type II contour $\mathfrak{L} = \mathfrak{L}^\alpha$, and interface line $\mathfrak{L}_{in} = \{(\theta,\,\varphi) : \theta = \theta_0,\,\forall\varphi\}$.

The dependence of eigenvalues μ_1^0 and μ_2^0 on κ is given on Fig. 1a and Fig. 1b, respectively, for type I of geometric data and for two variants of parameters: 1) $\alpha = 5/12$, $\delta = 1/6$, $\alpha_{in} = 7/12$, 2) $\alpha = 1/3$, $\delta = 1/6$, $\alpha_{in} = 1/2$. The graphs demonstrate considerable dependence of eigenvalues on κ.

For type II of geometric data with parameters $\alpha = 5/12$, $\theta_0 = 2/3$, $\kappa = 10$ the space views of the first $U_1^{0\,+}$ and the second $U_2^{0\,+}$ eigenfunctions with eigenvalues $\mu_1^0 = 0.090288$ and $\mu_2^0 = 1.453002$ are displayed on Fig. 2 and Fig. 3, respectively. The space views represent $2D$ graphs of the eigenfunctions, in which coordinates (θ,φ) are transformed by stereographic projection of the sphere $\mathbb{S}^2$ onto a plane (x_1, x_2), tangential to $\mathbb{S}^2$ at the North Pole.

4 The Solution of the Transmission BVP in Domain Ω

4.1 The Multipoles Ψ_k

In accordance with Theorem 2, all eigenvalues $\mu_n^m(M)$ can be enumerated as $\mu(k)$, $k = 1, 2, \ldots$, in order of their nondecreasing; each multiple eigenvalue should be counted according to its multiplicity. Thus, there arises respective numeration of the eigenfunctions $U_n^{m\,\pm}\big(\theta,\,\varphi\big)$ as

$U(\mu(k);\, \theta,\, \varphi)$ and, as a consequence, respective numeration of the multipoles $\Psi_k(r,\, \theta,\, \varphi)$; this manner of their numeration had already appeared in (3.1).

If our cone $\mathcal{K}$ is in fact a polyhedral angle, then a suitable representation for the Multipoles can be given. In order to formulate this representation let introduce a new system of spherical co–ordinates (r, Θ, Φ) related to an edge of the polyhedral angle.

Let us select a particular edge and define new Cartesian co–ordinates X, Y, Z with their origin at the vertex $\{0\}$ of the polyhedral angle disposed in such a way that the selected edge lies on axis Z, axis X lies on a face (or its extension), and axis Y is perpendicular to this face and is directed inside domain $\mathcal{K}$. Radial co–ordinate in the new system (r, Θ, Φ) coincides with the above one, and angle co–ordinates are defined by the standard formulas $\Phi = \arctan(Y/X)$, $\Theta = \arccos(Z/r)$. Denote the relation between old and new spherical co–ordinates by $\theta = \theta(\Theta,\, \Phi)$, $\varphi = \varphi(\Theta,\, \Phi)$. Then the desired representation for $U_k\,(\theta,\, \varphi) = V_k\,(\Theta,\, \Phi)$ has the form

$$V_k\,(\Theta,\, \Phi) \;=\; D_k^m\; \mathrm{P}_{\mu_k}^{-m/\beta}(\cos\Theta)\,\sin\frac{m\,\Phi}{\beta}\,. \tag{4.1}$$

Coefficients D_k^m in (4.1) can be computed as an integral over any curve $\{\Theta = \Theta_0 = \mathrm{const}\} \subset \mathcal{S}_{ex}$:

$$D_k^m \;= 2\left[\pi\,\beta\,\mathrm{P}_{\mu_k}^{-m/\beta}(\cos\Theta_0)\right]^{-1}\int_0^{\pi\,\beta} U_k\,(\theta,\, \varphi)\,\sin\frac{m\,\Phi}{\beta}\,d\Phi,$$

where $\theta = \theta\,(\Theta_0,\, \Phi),\; \varphi = \varphi\,(\Theta_0,\, \Phi)$.

4.2 The method of solving BVP

Now we turn to the transmission BVP (2.3)–(2.5) in domain Ω with cones of arbitrary base as described in Sect. 2. Note that $\partial\Omega$ and γ_{in} may have at most a finite number of edges and conical points. Since the boundary $\partial\Omega$ is Lipschitz, a Sobolev space $\overset{\circ}{W}{}_2^1(\mathcal{D})$ is defined habitually as a subspace of $W_2^1(\mathcal{D})$ consisting of functions having zero trace on $\partial\mathcal{D}$. Obviously, the space $\overset{\circ}{W}{}_2^1(\mathcal{D})$ is a Hilbert space with the inner product

$$[u,\, v;\; \overset{\circ}{W}{}_2^1(\mathcal{D})] = \int_{\mathcal{D}} u\, v\, ds \,+\, \int_{\mathcal{D}} (\nabla_\Gamma\, u,\, \nabla_\Gamma\, v)\, ds\,,$$

where ∇_Γ stands for a tangential component to Γ of the gradient ∇. In the following theorem, notation $W_2^{3/2}(\Omega)$ stands for the Sobolev — Slobodetskii space with the norm, where standard notations are used (see, e.g. [21], [23], [26]),

$$\|\psi; W_2^{3/2}(\Omega)\|^2 = \|\psi; W_2^1(\Omega)\|^2 + \sum_{|\alpha|=1}\int_{\Omega\,\times\,\Omega}\frac{|\mathcal{D}_x^\alpha\;\psi(x) - \mathcal{D}_y^\alpha\;\psi(y)|^2}{|x\,-\,y|^4}\,dx\,dy\,.$$

Theorem 3. *Let $h_{\mathcal{D}} \in \overset{\circ}{W}{}_2^1(\mathcal{D})$ and $h_{\mathcal{N}} \in L_2(\mathcal{N})$. Then the generalized solution $\psi \in \overset{\circ}{W}{}_2^1(\Omega,\, \gamma)$ in Theorem 1 belongs to $W_2^{3/2}(\Omega)$, and*

$$\|\psi;\; W_2^{3/2}\,(\Omega)\| \,\le\, C\left(\,\|h_{\mathcal{D}};\; \overset{\circ}{W}{}_2^1(\mathcal{D})\| \,+\, \|h_{\mathcal{N}};\; L_2(\mathcal{N})\|\,\right)$$

with constant $C > 0$ depending only on Ω.

Due to the embedding $W_2^{3/2}(\Omega)$ into $W_2^1(\partial\Omega)$ the trace on $\partial\Omega$ of the generalized solution $\psi \in W_2^{3/2}(\Omega)$ in Theorem 3 belongs to $W_2^1(\partial\Omega)$. Denote by $H(\Gamma)$ a space of all generalized solutions $\psi \in \overset{\circ}{W}{}_2^1(\Omega,\gamma) \cap W_2^{3/2}(\Omega)$ in Theorem 3 with boundary data $h_{\mathcal{D}} \in \overset{\circ}{W}{}_2^1(\mathcal{D})$ and $h_{\mathcal{N}} \in L_2(\mathcal{N})$. Clearly, Theorem 3 implies that $H(\Gamma)$ is a Hilbert space with the inner product

$$[u,\,v]_H = \int_{\mathcal{D}} uv ds + \int_{\mathcal{D}} (\nabla_\Gamma u,\, \nabla_\Gamma v) ds + \int_{\mathcal{N}} \partial_\nu u \partial_\nu v\, ds.$$

For $u \in W_2^{3/2}(\Omega)$, existence of the trace $\nabla_\Gamma u \in L_2(\mathcal{D})$ is guaranteed by the embedding of $W_2^{3/2}(\Omega)$ into $W_2^1(\partial\Omega)$. Notice that existence of trace $\partial_\nu u \in L_2(\mathcal{N})$ is guaranteed only for the functions $u \in H(\Gamma)$ by virtue of Theorem 3.

For basic functions $\{\Psi_k\}$ constructed in Sect. 3 it holds

Theorem 4. *The traces on Γ of the basic functions $\{\Psi_k\}$ form a complete system in $H(\Gamma)$ which is minimal.*

Proof of the completeness in Theorem 4 is based on the approximation theorems by F. Browder [29] for solutions of elliptic PDEs. Theorems [29] can be readily modified to include homogeneous boundary conditions on some part of the boundary.

A Cartesian product $\mathcal{H}(\mathcal{D},\mathcal{N}) \overset{\text{def}}{=} \overset{\circ}{W}{}_2^1(\mathcal{D}) \times L_2(\mathcal{N})$ consisting of ordered pairs $\{a_{\mathcal{D}},\, a_{\mathcal{N}}\}$, $a_{\mathcal{D}} \in \overset{\circ}{W}{}_2^1(\mathcal{D})$, $a_{\mathcal{N}} \in L_2(\mathcal{N})$, is a Hilbert space with the inner product

$$\big[\{a_{\mathcal{D}},\, a_{\mathcal{N}}\},\, \{b_{\mathcal{D}},\, b_{\mathcal{N}}\}\big]_{\mathcal{H}} = \int_{\mathcal{D}} a_{\mathcal{D}} b_{\mathcal{D}} ds + \int_{\mathcal{D}} (\nabla_\Gamma a_{\mathcal{D}}\,,\, \nabla_\Gamma b_{\mathcal{D}}) ds + \int_{\mathcal{N}} a_{\mathcal{N}} b_{\mathcal{N}} ds$$

which induces the norm

$$\|a_{\mathcal{D}},\, a_{\mathcal{N}}\|^2_{\mathcal{H}} = \int_{\mathcal{D}} |a_{\mathcal{D}}|^2 ds + \int_{\mathcal{D}} |\nabla_\Gamma a_{\mathcal{D}}|^2 ds + \int_{\mathcal{N}} |a_{\mathcal{N}}|^2\, ds.$$

Let $L : H(\Gamma) \to \mathcal{H}(\mathcal{D},\,\mathcal{N})$ be a linear operator defined as $L\psi = \big\{\psi\big|_{\mathcal{D}},\ \partial_\nu\psi\big|_{\mathcal{N}}\big\}\ \forall\,\psi \in H(\Gamma)$. From Theorem 3 it follows

Corollary 1. *The linear operator L is an isometry of $H(\Gamma)$ onto $\mathcal{H}(\mathcal{D},\,\mathcal{N})$.*

For the basic functions $\{\Psi_k\}$, from Corollary 1 and Theorem 4 follows

Corollary 2. *The system $\{L\,\Psi_k\}$ is complete and minimal in $\mathcal{H}(\mathcal{D},\,\mathcal{N})$.*

Applying Corollary 2, we approximate the solution $\psi(r,\,\theta,\,\varphi)$ of the BVP (2.3)–(2.5) by a sequence $\{\psi^{(N)}(r,\theta,\varphi)\}$ of linear combinations with respect to the first N basic functions Ψ_k:

$$\psi(r,\theta,\varphi) = \lim_{N\to\infty} \psi^{(N)}(r,\theta,\varphi), \qquad \psi^{(N)}(r,\theta,\varphi) = \sum\nolimits_{k=1}^{N} Q_k^{(N)}\, \Psi_k(r,\,\theta,\,\varphi). \tag{4.2}$$

Here coefficients $Q_k^{(N)}$ are to be found using the condition of the least square deviation of the approximate solution $\psi^{(N)}$ from the boundary function $h = \{h_{\mathcal{D}}, h_{\mathcal{N}}\} \in \mathcal{H}(\mathcal{D},\mathcal{N})$ corresponding to (2.6) on Γ: $\|L\psi^{(N)} - h\|_{\mathcal{H}} \to \min$. This condition leads to the following system of linear equations with respect to the unknown coefficients $Q_k^{(N)}$, where $l = 1,\, 2\, \ldots,\, N$:

$$\sum\nolimits_{k=1}^{N} Q_k^{(N)}\, G_k^l = h^l\,, \qquad G_k^l = [L\Psi_k,\, L\Psi_l]_{\mathcal{H}}\,, \quad h^l = [h,\, L\Psi_l]_{\mathcal{H}}\,.$$

The method of least squares guarantees the convergence of the sequence $L\psi^{(N)}$ in the Hilbert space $\mathcal{H}(\mathcal{D},\mathcal{N})$, whence by Corollary 1 follows the convergence of the sequence $\psi^{(N)}$ in the Hilbert space $H(\Gamma)$. Now for the sequence of approximate solutions $\{\psi^{(N)}\}$, reference to Theorem 3 completes the proof of its convergence in $W_2^{3/2}(\Omega)$ to the exact solution $\psi \in \overset{\circ}{W}{}_2^1(\Omega,\gamma)\cap$ $\cap W_2^{3/2}(\Omega)$.

4.3 Asymptotics near the edges

Turn again to the selected edge mentioned in Sect. 4.1. Introduce a cylindrical system of co–ordinates related to this edge by the use of the Cartesian X, Y, Z and the spherical (r,Θ,Φ) co–ordinate systems defined in Sect. 4.1. Namely, let Z be the co–ordinate from the above Cartesian system, Φ the co–ordinate from the above spherical system, and ϱ is defined by the formula $\varrho = \sqrt{r^2 - Z^2}$. Then the desired cylindrical co–ordinate system is (ϱ, Z, Φ).

Starting from the view (4.2) of the solution and using representation (4.1) for the multipoles we derive an asymptotics for the solution of the BVP near the edge with dihedral angle of value $\pi\beta$ when $\varrho \to 0,\ \ Z \to 0$:

$$\Psi \sim \varrho^{1/\beta}\sin\frac{\Phi}{\beta}\left[\mathcal{I}_{1,1}\, Z^{\mu_1 - 1/\beta} + \ldots\right] + \varrho^{2/\beta}\sin\frac{2\Phi}{\beta}\left[\mathcal{I}_{2,1}\, Z^{\mu_2 - 1/\beta} + \ldots\right] + \ldots .$$

Quantities $\mathcal{I}_{1,1}$ and $\mathcal{I}_{2,1}$ appearing here can be expressed via coefficients of expansions (4.1), (4.2), in particular $\mathcal{I}_{1,1} = 2^{-1/\beta}\,[\Gamma(1 + 1/\beta)]^{-1}\, Q_1\, D_1^1$, where $\Gamma(x)$ is Gamma–function [27].

Note that coefficients Q_k^n in expansion (4.2) are named intensity factors at the vertex of the cone (polyhedral angle) and quantities $\mathcal{I}_{1,1}$, $\mathcal{I}_{2,1}$ the intensity factors at its edge. From what was said it follows that our method provides computation of all mentioned intensity factors along with the solution itself.

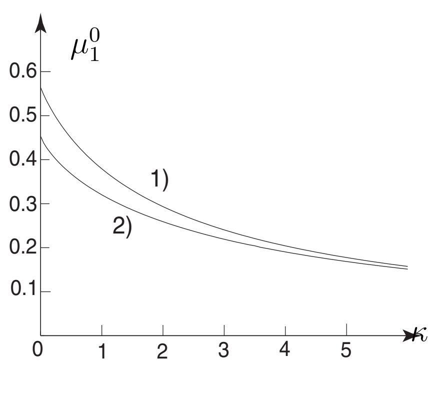

Fig. 1a.

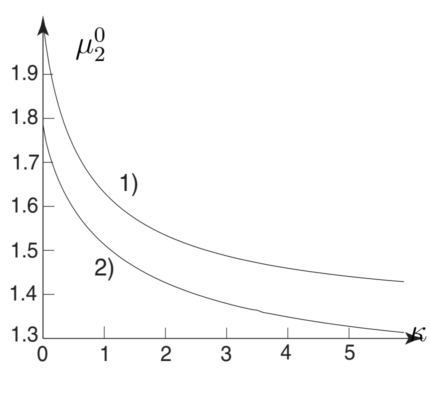

Fig. 1b.

Fig. 1. Dependence of eigenvalues μ_1^0 and μ_2^0 on κ for type I of geometric data and for two variants of parameters: 1) $\alpha = 5/12$, $\delta = 1/6$, $\alpha_{in} = 7/12$, 2) $\alpha = 1/3$, $\delta = 1/6$, $\alpha_{in} = 1/2$.

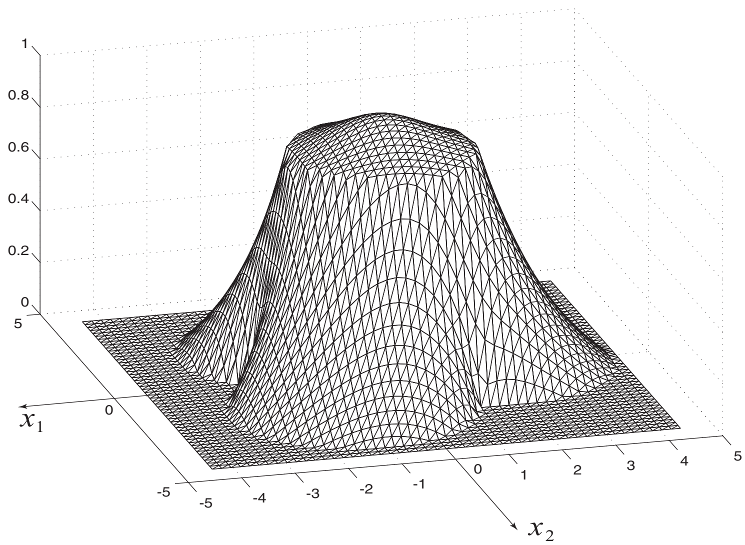

Fig. 2. Space view of the first eigenfunction U_1^{0+} with parameters $\alpha = 5/12$, $\theta_0 = 2/3$, $\kappa = 10$ and eigenvalue $\mu_1^0 = 0.090288$.

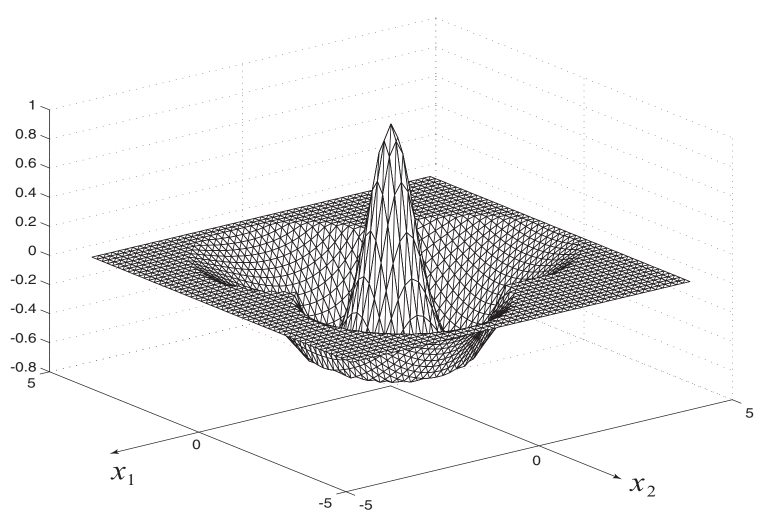

Fig. 3. Space view of the second eigenfunction U_2^{0+} with parameters $\alpha = 5/12$, $\theta_0 = 2/3$, $\kappa = 10$ and eigenvalue $\mu_2^0 = 1.453002$.

Bibliography

1. S. Nicaise, A.-M. Sändig. Transmission problems for the Laplace and elasticity operators: regularity and boundary integral formulation // Mathem. Models and Meth. in Appl. Sci. V. 9, №6. 1999. P. 855-898.

2. V.A. Kondrat'ev. Boundary value problems for elliptic equations in domains with conical or angular points // Trudy Moskovskogo Mat. Obschestva. V. 16. 1967. P. 209-292. (Trans. Moscow Math. Soc. V. 16. 1967. P. 227-301.)

3. V.A. Kondrat'ev, O.A. Oleinik. Boundary value problems for partial differential equations in nonsmooth domains // Uspekhi Mat. Nauk. V. 38. 1983. P. 3-76. Trans. Russian Math. Survey. V. 38. 1983. P. 1-86.

4. M. Dauge. Elliptic boundary value problems in corner domains — smoothness and asymptotics of solutions. Lect. Notes Math. Heidelberg: Springer, 1988.

5. A.E. Beagles, J.R. Whiteman. General conical singularities in three–dimensional Poisson problems // Math. Meth. Appl. Sci. V. 11. 1989. P. 215-235.

6. S.A. Nazarov, B.A. Plamenevskij. Elliptic problems in domains with pieswise smooth boundaries, Moscow: Nauka, 1991.

7. P. Grisvard. Elliptic problems in nonsmooth domains, London: Pitman, 1985.

8. P. Grisvard. Singularities in boundary value problems. Res. Notes Appl. Math., Paris: Masson, Bonn: Springer, 1992.

9. H. Blum. Numerical treatment of corner and crack singularities, In Finite element and boundary element techniques from mathematical and engineering point of view (Edited by E. Stein and W.L. Wendland). CISM Courses and Lectures (301). Vienna: Springer–Verlag. 1988. P. 171–212.

10. E. Stephan, J.R. Whiteman. Singularities of the Laplacian at corners and edges of three–dimensional domains and their treatment with finite element methods // Math. Meth. Appl. Sci. V. 10. 1988. P. 339-350.

11. H. Schmitz, K. Volk, W. Wendland. Three–dimensional singularities of elastic fields near vertices // Numer. Methods Partial Differ. Equations. V. 9. 1993. P. 323-337.

12. J.M.–S. Lubuma, S. Nicaise. Dirichlet problems in polyhedral domains II: Approximation by FEM and BEM // J. Comput. and Appl. Math. V. 61. 1995. P. 13-27.

13. Th. Apel, A.-M. Sändig, J.R. Whiteman. Graded mesh refinement and error estimates for finite element solutions of elliptic boundary value problems in non–smooth domains // Math. Meth. Appl. Sci. V. 19. 1996. P. 63-85.

14. V.I. Vlasov. On a method of solving some mixed planar problems for the Laplace equation // Dokl. Akad. Nauk SSSR. V. 237, №5. 1977. P. 1012-1015. Trans. Soviet Math. Dokl. V. 237. №5. 1977.

15. V.I. Vlasov. Boundary value problems in domains with curved boundary, Moscow: Computing Center Russian Acad. Sci. 1987. (In Russian).

16. V.I. Vlasov. Multipole method for solving some boundary value problems in complex-shaped domains // Zeitschr. Angew. Math. Mech. V. 76, Suppl. 1. 1996. P. 279-282.

17. V.I. Vlasov. A meshless method for solving boundary value problems in 3D domains of complex shape // The Fourth International Congress on Industrial and Applied Mathematics, Edinburgh, 5–9 July 1999. Book of Abstracts. P. 323.

18. V.I. Vlasov. A method of solving boundary value problems for the Laplace equation in domains with cones // Russin Doklady (Doklady Akademii Nauk). V. 397, №5. 2004. P. 586-589. (In Russian.)

19. S.L. Skorokhodov, V.I. Vlasov. The Multipole method for the Laplace equation in domains with polyhedral corners // Comp. Assisted Mechan. and Engin. Scien. V. 11. 2004. P. 223-238.

20. M. Brelo. Elements of the classical theory of the potential, Moscow: Mir. 1964.

21. O.A. Ladyzhenskaya, N.N. Ural'tseva. Linear and quasilinear equations of elliptic type, Moscow: Nauka. 1964.

22. J.–L. Lions, E. Magenes. Problèmes aux limites non homogènes et applicasions, Paris: Dunod. 1968.

23. E.M. Stein. Singular integrals and differentiability properties of functions, Princeton: University Press. 1970.

24. B.V. Palcev. On the mixed problem with nonhomogeneous boundary conditions for elliptic equations of the second order in Lipschits domains // Matematicheskii Sbornik. V. 187, 1996. P. 59-116. (In Russian.)

25. L.N. Slobodetskii. Generalised spaces of S.L.Sobolev and their applications to boundary value problems for partial differential equations // Uchenye Zapiski Leningradskogo Gosud. Pedag. Inst. V. 197, 1958. P. 54-112. (In Russian.)

26. R. Adams. Sobolev spaces, New York: Acad. Press. 1975.

27. H. Bateman, A. Erdelyi. Higher transcendental functions, New York: Mc Graw–Hill Co. 1953.

28. Yu.L. Luke. Mathematical functions and their approximations, New York: Academic Press. 1975.

29. F.E. Browder. Function analysis and partial differential equations, 2 // Math. Ann. V. 145. 1962. P. 81-226.

МЕТОД МУЛЬТИПОЛЕЙ ДЛЯ НЕКОТОРЫХ ЭЛЛИПТИЧЕСКИХ КРАЕВЫХ ЗАДАЧ С РАЗРЫВНЫМ КОЭФФИЦИЕНТОМ

С.Л. Скороходов, В.И. Власов

Вычислительный центр им. А.А.Дородницына РАН,

ул. Вавилова, 40, Москва, 119991, Россия e-mail: skor@ccas.ru,vlasov@ccas.ru

Аннотация. Разработан аналитико-численный метод решения краевых задач в пространственных областях с конусами произвольного основания для эллиптического уравнения с кусочно-постоянным коэффициентом. Решение задачи находится с использованием специальных базисных функций – мультиполей, которые строятся в явном виде. Метод обеспечивает высокоточное вычисление решения, его производных, показателей сингулярности и коэффициентов интенсивности вблизи геометрических особенностей – ребер и вершины конуса.

Ключевые слова: краевые задачи, области с конусами, метод мультиполей, показатели сингулярности, коэффициенты интенсивности.

УДК 517.9

GENERALIZED POTENTIALS OF DOUBLE LAYER FOR SECOND ORDER ELLIPTIC SYSTEMS

A.P.Soldatov
Belgorod State University
Pobedy str., 85, Belgorod, 308015, Russia, e-mail: Soldatov@bsu.edu.ru

Abstract. Second order elliptic systems on the plane are considered. The notion of generalized potentials of double layer for these systems is introduced.

Keywords: second order elliptic systems, lame system, potentials of double layer, Dirichlet problem.

1 Second order elliptic systems

Let us consider the elliptic system of second order

$$\sum_{i,j=1}^{2} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} = 0, \quad u = (u_1, \dots, u_l), \quad x_1 = x,\, x_2 = y,$$

with constant and only leading coefficients $a_{ij} \in \mathbb{R}^{l\times l}$. In view of the elliptic condition

$$\det\Big(\sum a_{ij}\lambda_i\lambda_j\Big) \neq 0, \quad \lambda_1, \lambda_2 \in \mathbb{R},$$

the characteristical polynomial

$$\chi(z) = \det p(z), \quad p(z) = a_{11} + (a_{12} + a_{21})z + a_{22}z^2$$

has no real roots. Let σ_+ denote a set of all these roots in the upper half-plane.

Let $D \subseteq \mathbb{C}^2$ be a finite domain with a smooth boundary $\Gamma = \partial D$. As it's well known the Dirichlet problem

$$u\big|_\Gamma = f$$

isn't always Fredholm. The first example of this type belongs to A. V. Bitsadze[1]. He noticed that the homogeneous Dirichlet problem for elliptic system with coefficients ($l = 2$)

$$a_{11} = -a_{22} = 1, \quad a_{12} = a_{21} = \begin{pmatrix} 0 & \pm 1 \\ \mp 1 & 0 \end{pmatrix}$$

in the unite circle has infinitely linear independent solutions.

Later A. V. Bitsadze introduced the notion of the so-called weakly connected elliptic systems for which the Dirichlet problem is Fredholm. According to modern elliptic theory this requirement simply implies that the corresponding Shapiro- Lopatinski condition holds[2]. It's convenient to formulate this condition in the following way.

The elliptic system is weakly connected iff

The work was supported in part by the Russian Foundation of Basic Research (RFBR)(project No. 07-01-00299) and by the National Natural Science Foundation of China (NSFC) in the framework of the bilateral project "Complex Analysis and its applications"(project No. 08-01-92208-GFEN).

$$\det \left[\int_{\mathbb{R}} p^{-1}(\lambda) d\lambda \right] \neq 0.$$

The Bitsadze example stimulated the definitions of the various classes of elliptic systems for which the Dirichlet problem is Fredholm. The most important of them was the notion of strong elliptic system introduced by M. I. Vishik[3]. They are defined by the condition of positive definiteness of the matrix

$$\sum_{i,j=1}^{2} a_{ij}\lambda_i\lambda_j > 0$$

for all $\lambda_, \lambda_2 \in \mathbb{R}$, $|\lambda_1| + |\lambda_2| \neq 0$.

In this case the matrix $p^{-1}(\lambda)$ is also positive definite, so these systems are really weakly connected. More restrict condition was introduced earlier by C. Somigliano[4] and is expressed in the form

$$a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} > 0.$$

The intermediate position between these definitions occupies the notion of the strengthened elliptic system [5]. By definition this system have to be elliptic and the matrix $a \geq 0$. Note that another classification of elliptic systems in the case $l = 2$ is given by Lin Wei[6] and Wu Ci-Quian[7].

2 Generalized potentials of double layer

Let the elliptic system be weakly connected. As it will be said earlier then the Dirichlet problem is Fredholm. More exactly the following result is valid [8]. Here and below $C^{+0}(E)$ implies the Holder class $\cup_{\mu>0} C^{\mu}(E)$.

Let $\Gamma = \partial D$ be Lyapunov contour i.e. its inner normal $n(t) = n_1(t) + in_2(t) \in C^{+0}(\Gamma)$ and let $f \in C^{+0}(\Gamma)$. Then homogeneous Dirichlet problem has a finite number linear independent solutions $u_1, \ldots, u_n \in C^{+0}(\overline{D})$ and there exist a real vector- valued linear independent functions $g_1, \ldots, g_n \in C^{+0}(\Gamma)$ such that nonhomogeneous Dirichlet problem is solvable in $C^{+0}(\overline{D})$ iff

$$(f, g_i) = 0, \ 1 \leq i \leq n,$$

where

$$(f, g) = \int_{\Gamma} f(t)g(t)|dt|.$$

The case of strengthened elliptic system is remarkable as $n = 0$ for these systems. In other words *the Dirichlet problems for a strengthened elliptic system is uniquely solved.*

The main result of this talk is the following: *if $f \in C(\Gamma)$ satisfies the orthogonality conditions then the Dirichlet problem is solvable in the class $C(\overline{D})$.*

Our approach is based on using generalized potentials of double layer for the elliptic system. From the weakly connected property it follows the following lemma: *there exists the unique matrix $J \in \mathbb{C}^{l \times l}$ such that*

$$a_{11} + (a_{12} + a_{21})J + a_{22}J^2 = 0,$$

$$\sigma(J) = \sigma_+, \quad \det(\operatorname{Im} J) \neq 0.$$

Recall that σ_+ denotes a set of all roots in the upper half-plane of the characteristical polynomial $\chi(z) = \det p(z)$, $p(z) = a_{11} + (a_{12} + a_{21})z + a_{22}z^2$. The matrix J is called a characteristical matrix of the elliptic system. If it is diagonal then the system reduces to l scalar equations. More exactly the exists an invertible matrix c such that all matrixes ca_{ij} are diagonal. So we may suggest that J is not diagonal.

Let us put

$$Q(t,\xi) = \frac{n_1(t)\xi_1 + n_2(t)\xi_2}{|\xi|^2} H(\xi),$$

$$H(\xi) = \operatorname{Im}[(-\xi_2 1 + \xi_1 J)(\xi_1 1 + \xi_2 J)^{-1}],$$

where 1 implies the unit matrix and n is the unit vector of inner normal. *Then the integral*

$$(P\varphi)(z) = \frac{1}{\pi}\int_\Gamma Q(t, t-z)\varphi(t)|dt|, \quad z \in D,$$

describes solutions of the elliptic system. Note that for $H = 1$ this integral corresponds to the classical potentials of double layer for Laplace equation. The following theorem shows that $P\varphi$ plays an analogous role for the elliptic system.

The integral operator P is bounded $C(\Gamma) \to C(\overline{D})$ and

$$(P\varphi)^+(t_0) = \varphi(t_0) + \int_\Gamma Q(t, t-t_0)\varphi(t)|dt|, \quad t_0 \in \Gamma.$$

Let $K\varphi$ imply the integral on the right hand side. Under assumptions $n(t) \in C^{+0}(\Gamma)$ the kernel $k(t_0,t) = (t-t_0)Q(t,t-t_0)$ belongs to $C^{+0}(\Gamma\times\Gamma)$ and $k(t,t) \equiv 0$. So the operator K is compact in $C(\Gamma)$.

Theorem. *The exist a finite-dimensional space $X \subseteq C^{+0}(\overline{D})$ of solutions of the elliptic system and a space $Y \subseteq C^{+0}(\Gamma)$ of the same dimension such that each solution $u \in C(\overline{D})$ of the elliptic system is uniquely represented in the form*

$$u = P\varphi + u_0, \quad u_0 \in X,$$

where $\varphi \in C(\Gamma)$ satisfies the orthogonality condition $(\varphi, g) = 0$, $g \in Y$.

If the system is strengthened elliptic then in this representation $X = 0$, $Y = 0$.

The theorem shows that the Dirichlet problem is equivalent to the following system of Fredholm integral equations:

$$\varphi + K\varphi + \sum_1^m \lambda_i u_i = f,$$

$$(\varphi, g_i) = 0, \quad i = 1, \ldots, m,$$

where $u_1, \ldots, u_m$ and $g_1, \ldots, g_m$ are basises of X and Y respectively.

In the case $l = 2$ the matrix $H(\xi)$ can be described explicitly. In this case there are only two possibility for σ_+ when (i) $\sigma_+ = \{\nu_1, \nu_2\}$, $\nu_1 \neq \nu_2$, and (ii) $\sigma_+ = \{\nu\}$. So the exists an invertible matrix $b \in \mathbb{C}^{2\times 2}$ such that

$$(i)\; bJb^{-1} = \begin{pmatrix} \nu_1 & 0 \\ 0 & \nu_2 \end{pmatrix}, \quad (ii)\; bJb^{-1} = \begin{pmatrix} \nu & 1 \\ 0 & \nu \end{pmatrix}.$$

The case $bJb^{-1} = \nu$ is excluded as the matrix J is not diagonal. Note that the matrixes

$$E_1 = b \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} b^{-1}, \quad E_2 = b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} b^{-1},$$

don't depend on the choice of b.

In this terms we have:

$$(i)\ H(\xi) = (\mathrm{Im}^2 \nu_2) g(\xi, \nu_2) + \mathrm{Im}[(\nu_1 - \nu_2) g(\xi, \nu_1) g(\xi, \nu_2) E_1],$$

$$(ii)\ H(\xi) = (\mathrm{Im}^2 \nu) g(\xi, \nu) + \mathrm{Im}[g^2(\xi, \nu) E_2],$$

where $g(\xi, \nu) = |\xi| (\xi_1 + \nu \xi_2)^{-1}$.

3 Applications to the plane elasticity

The plane elastic medium is characterized by the displacement vector $u = (u_1, u_2)$ and by stress and deformation tensors

$$\sigma = \begin{pmatrix} \sigma_1 & \sigma_3 \\ \sigma_3 & \sigma_2 \end{pmatrix}, \quad \varepsilon = \begin{pmatrix} \varepsilon_1 & \varepsilon_3 \\ \varepsilon_3 & \varepsilon_2 \end{pmatrix},$$

where

$$\varepsilon_i = \frac{\partial u_i}{\partial x_i}, \quad i = 1, 2, \quad 2\varepsilon_3 = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_1}{\partial x_2}.$$

They are connected by Hooke law i.e. by linear relation

$$\tilde\sigma = \alpha \tilde\varepsilon, \quad \alpha = \begin{pmatrix} \alpha_1 & \alpha_4 & \alpha_5 \\ \alpha_4 & \alpha_2 & \alpha_6 \\ \alpha_5 & \alpha_6 & \alpha_3 \end{pmatrix} > 0,$$

where $\tilde\sigma = (\sigma_1, \sigma_2, \sigma_3)$, $\tilde\varepsilon = (\varepsilon_1, \varepsilon_2, 2\varepsilon_3)$.

If the external forces are absent then the equilibrium equations have the form

$$\frac{\partial \sigma_{(1)}}{\partial x_1} + \frac{\partial \sigma_{(2)}}{\partial x_2} = 0,$$

where $\sigma_{(j)}$ means $j-$column of the matrix σ. Using the Hooke law we receive the Lame system

$$a_{11} \frac{\partial^2 u}{\partial x^2} + (a_{12} + a_{21}) \frac{\partial^2 u}{\partial x \partial y} + a_{22} \frac{\partial^2 u}{\partial y^2} = 0$$

for the replacement vector u with the coefficients a_{ij}, defined by the matrix

$$a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \alpha_1 & \alpha_6 & \alpha_6 & \alpha_4 \\ \alpha_6 & \alpha_3 & \alpha_3 & \alpha_5 \\ \alpha_6 & \alpha_3 & \alpha_3 & \alpha_5 \\ \alpha_4 & \alpha_5 & \alpha_5 & \alpha_2 \end{pmatrix}.$$

This system is strengthened elliptic and rang $a = 3$.

The elastic medium is called orthotropic if $\alpha_5 = \alpha_6 = 0, \quad \alpha_3 + \alpha_4 \neq 0$, and isotropic if $\alpha_5 = \alpha_6 = 0, \quad \alpha_1 = \alpha_2 = 2\alpha_3 + \alpha_4$. We can also point out the special case $\alpha_5 = \alpha_6 = 0, \quad \alpha_3 + \alpha_4 = 0$. In this case the Lame system reduces to scalar equations

$$\alpha_1 \frac{\partial^2 u_1}{\partial x^2} + \alpha_3 \frac{\partial^2 u_1}{\partial y^2} = 0, \quad \alpha_3 \frac{\partial^2 u_2}{\partial x^2} + \alpha_2 \frac{\partial^2 u_2}{\partial y^2} = 0.$$

So this case we put away below.

Let us consider the characteristic polynomial of Lame system

$$p(z) = a_{11} + (a_{12} + a_{21})z + a_{22}z^2 = \begin{pmatrix} p_1 & p_3 \\ p_3 & p_2 \end{pmatrix},$$

where $p_1(z) = \alpha_1 + 2\alpha_6 z + \alpha_3 z^2$, $p_2(z) = \alpha_3 + 2\alpha_5 z + \alpha_2 z^2$, $p_3(z) = \alpha_6 + (\alpha_3 + \alpha_4)z + \alpha_5 z^2$.

In the case (i) we can put

$$E_1 = \frac{1}{p_2(\nu_2)p_3(\nu_1) - p_2(\nu_1)p_3(\nu_2)} \begin{pmatrix} -p_2(\nu_1)p_3(\nu_2) & -p_2(\nu_1)p_2(\nu_2) \\ -p_3(\nu_1)p_3(\nu_2) & p_2(\nu_2)p_3(\nu_1) \end{pmatrix},$$

if one of the following conditions $(*)$

$$\alpha_3^2 < \alpha_1\alpha_2, \ \alpha_5^2 < \alpha_2\alpha_3, \ \alpha_2\alpha_6 = \alpha_3\alpha_5, \ \alpha_2(\alpha_3 + \alpha_4) = 2\alpha_5^2,$$

disturbs and

$$E_1 = \frac{1}{p_1(\nu_1)p_3(\nu_2) - p_1(\nu_2)p_3(\nu_2)} \begin{pmatrix} -p_1(\nu_2)p_3(\nu_1) & -p_3(\nu_1)p_3(\nu_2) \\ -p_1(\nu_1)p_1(\nu_2) & p_1(\nu_1)p_3(\nu_2) \end{pmatrix},$$

if one of the following conditions $(**)$

$$\alpha_3^2 < \alpha_1\alpha_2, \ \alpha_6^2 < \alpha_1\alpha_3, \ \alpha_1\alpha_5 = \alpha_3\alpha_6, \ \alpha_1(\alpha_3 + \alpha_4) = 2\alpha_6^2$$

disturbs.

In the case (ii) we can put

$$E_2 = \frac{1}{p_2'(\nu)p_3(\nu) - p_2(\nu)p_3'(\nu)} \begin{pmatrix} p_2(\nu)p_3(\nu) & p_2^2(\nu) \\ -p_3^2(\nu) & -p_2(\nu)p_3(\nu) \end{pmatrix}.$$

Note that fulfilments of both conditions $(*)$ and $(**)$ is equivalent to the special case $\alpha_5 = \alpha_6 = 0, \ \alpha_3 + \alpha_4 = 0$ when the Lame system is diagonal.

In the orthotropic case the polynomial p_j are simplify:

$$p_1(z) = \alpha_1 + \alpha_3 z^2, \ p_2(z) = \alpha_3 + \alpha_2 z^2, \ p_3(z) = (\alpha_3 + \alpha_4)z,$$

so in this case

$$E_1 = \frac{(\alpha_3 + \alpha_4)^{-1}}{\nu_1 p_2(\nu_2) - \nu_2 p_2(\nu_1)} \begin{pmatrix} -p_2(\nu_1)(\alpha_3 + \alpha_4)\nu_2 & -p_2(\nu_1)p_2(\nu_2) \\ -(\alpha_3 + \alpha_4)^2\nu_1\nu_2 & p_2(\nu_2)(\alpha_3 + \alpha_4)\nu_1 \end{pmatrix},$$

$$E_1 = \frac{(\alpha_3 + \alpha_4)^{-1}}{\nu_2 p_1(\nu_1) - \nu_1 p_1(\nu_2)} \begin{pmatrix} -p_1(\nu_2)(\alpha_3 + \alpha_4)\nu_1 & -(\alpha_3 + \alpha_4)^2\nu_1\nu_2 \\ -p_1(\nu_1)p_1(\nu_2) & p_1(\nu_1)(\alpha_3 + \alpha_4)\nu_2 \end{pmatrix},$$

respectively to $(*)$, $(**)$ and

$$E_2 = \frac{1}{\alpha_3\nu^2-\alpha_2}\begin{pmatrix} \nu(\alpha_2+\alpha_3\nu^2) & (\alpha_3+\alpha_4)^{-1}(\alpha_2+\alpha_3\nu^2)^2 \\ -(\alpha_3+\alpha_4)\nu^2 & -\nu(\alpha_2+\alpha_3\nu^2) \end{pmatrix}.$$

Especially the simple picture we have in the orthotropic case when $\nu = i$ and $\alpha_1 = \alpha_2 = 2\alpha_3+\alpha_4$. In this case

$$E_2 = -\frac{1}{æ}\begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}, \quad æ = \frac{\alpha_1+\alpha_3}{\alpha_1-\alpha_3},$$

and therefor

$$H(\xi) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{æ|\xi|^2}\begin{pmatrix} \xi_2^2-\xi_1^2 & 2\xi_1\xi_2 \\ 2\xi_1\xi_2 & \xi_1^2-\xi_2^2 \end{pmatrix}.$$

Another function theoretical approaches for orthotropic Lame system were suggested by R P.Gilbert[9, 10].

Bibliography

1. А.В. Бицадзе. О единственности решения задачи Дирихле для эллиптических уравнений с частными производными. Успехи матем. наук, 1948. 3, N 6, 153-154.

2. С.А. Назаров, Б.А. Пламеневский. Эллиптические задачи в областях с кусочно гладкой границей. М., Наука, 1991, 336 с.

3. M.I. Vishik. On strong elliptic systems of differential equations, Matematicheskiy Sbornik, 1951. 29, 615-676, in Russian.

4. C. Somigliana. Sui sisteme simmetrici di equazioni a derivate parziali, Ann. Math. Pure et Appl., 1894. II, v. 22, 143-156

5. А.П. Солдатов, С.П. Митин. Об одном классе сильно эллиптических систем Дифференц.уравн. 1997. Т.33, N 8. С.1118–1122.

6. H. Begehr, Lin Wei. A mixed–contact problem in orthotropic elasticity, in Partial diifferential equations with Real Analysis, H. Begerhand, A.Jeffrey, eds., Longman Scientific & Technical, 1992. 219-239.

7. Hua Luo-Keng, Wu Ci-Quian, Lin Wei. Linear systems of partial differencial equations of second with constant coefficients, Science Press, Beijing, 1979

8. А.В. Бицадзе. Краевые задачи для эллиптических уравнений второго порядка. М.,Наука, 1966.

9. R.P. Gilbert Plane ellipticity and related problems, Amer. Math. Soc., Providence, Rl, 1981.

10. R.P. Gilbert, Lin Wei. Function theoretical solutions to problems of orthotropic elasticity, J. Elasticity, 1985. 15, 143-154,

ОБОБЩЕННЫЙ ПОТЕНЦИАЛ ДВОЙНОГО СЛОЯ ДЛЯ ЭЛЛИПТИЧЕСКИХ СИСТЕМ ВТОРОГО ПОРЯДКА

А.П. Солдатов

Белгородский государственный университет,

ул. Победы, 85, Белгород, 408015, Россия e-mail: Soldatov@bsu.edu.ru

Аннотация. Рассматриваются эллиптические слабо связанные (по терминологии А.В. Бицадзе) системы второго порядка с постоянными (и только старшими) коэффициентами. Для этих систем вводится понятие потенциалов двойного слоя, не связанное с фундаментальным решением. Оно позволяет редуциовать задачу Дирихле к эквивалентной системе интегральных уравнений Фредгольма на границе области.

Ключевые слова: эллиптические системы второго порядка, системы Ламэ, потенциал двойного слоя, задача Дирихле.

2000 MSC 32A15, 32A20, 32A22, 32H30

SHARING SET AND NORMAL FUNCTION OF HOLOMORPHIC FUNCTIONS

Jun-Feng Xu
Department of Mathematics, Wuyi University,
Jiangmen, Guangdong 529020, China, e-mail: lvfeng@mail.sdu.edu.cn

Abstract. In this paper, we use the idea of sharing set to prove: Let $\mathcal{F}$ be a family of holomorphic functions in the unit disc, a_1 and a_2 be two distinct finite numbers and $a_1 + a_2 \neq 0$. If for any $f \in \mathcal{F}$, $E_f(S) = E_{f'}(S), S = \{a_1, a_2\}$, in the unit disc, then f is an α-normal function.

Keywords: entire functions, uniqueness, Nevanlinna theory, normal family.

1 Introduction and main results

Let D be a domain in $\mathbb{C}$ and let $\mathcal{F}$ be a family of meromorphic functions defined in D. The family $\mathcal{F}$ is said to be normal in D, in the sense of Montel, if each sequence $\{f_n\} \subset \mathcal{F}$ contains a subsequence $\{f_{n_j}\}$ that converges, spherically locally uniformly in D, to a meromorphic function or to ∞.(see. [10])

In this paper, we assume that f, g are two meromorphic functions on D and S_1, S_2 are two sets. We denote $\overline{E}_f(S_1) \subset \overline{E}_g(S_2)$ by $f(z) \in S_1 \Rightarrow g(z) \in S_2$. If $\overline{E}_f(S_1) = \overline{E}_g(S_2)$, we denote this condition by $f(z) \in S_1 \Leftrightarrow g(z) \in S_2$. Similarly, if $E_f(S_1) = E_g(S_2)$, we denote this condition by $f(z) \in S_1 \rightleftharpoons g(z) \in S_2$. If the set S has only one element, say a, we denote $f(z) \in S$ by $f(z) = a$ (see [15]).

Schwick[14] was the first to draw a connection between values shared by functions in $\mathcal{F}$ (and their derivatives) and the normality of the family $\mathcal{F}$. Specially, he showed that if there exist three distinct complex numbers a_1, a_2, a_3 such that f and f' share $a_j (j = 1, 2, 3)$ in D for each $f \in \mathcal{F}$, then $\mathcal{F}$ is normal in D. Pang and Zalcman [9] extended this result as follows.

Theorem A. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, and let a, b, c, d be complex numbers such that $c \neq a$ and $d \neq b$. If for each $f \in \mathcal{F}$ we have $f(z) = a \Leftrightarrow f'(z) = b$ and $f(z) = c \Leftrightarrow f'(z) = d$, then $\mathcal{F}$ is normal in D.*

Definition 1.1 (see. [6, 7]) *A meromorphic function f is a normal function in the unit disc D if and only if there exists a constant $C(f)$ (which depends on f) such that*

$$(1 - |z|^2) f^\sharp(z) < C(f),$$

where $f^\sharp(z) = |g'(z)|/(1 + |g(z)|^2)$ is the spherical derivative of f.

In 2000, X.C. Pang [8] considered the normal function by using the condition of share values.

Theorem B. *Let $\mathcal{F}$ be a family of meromrophic functions in the unit disc, a_1, a_2 and a_3 be three distinct finite numbers. If for any $f \in \mathcal{F}$,*

$$\overline{E}_f(a_i) = \overline{E}_{f'}(a_i), \quad i = 1, 2, 3,$$

The author was supported by the NSF of China (10771121), the NSF of Guangdong Province (9452902001003278) and Excellent Young Fund of Department of Education of Guangdong (LYM08097).

in the unit disc, then there exists a positive M, such that for every $f \in \mathcal{F}$, we have

$$(1-|z|^2)f^\sharp(z) < M,$$

where M depends on a_1, a_2 and a_3.

In fact, from the proof of Theorem B, one can get the following corollary.

Corollary 1.2 *Let $\mathcal{F}$ be a family of holomorphic functions in the unit disc, a_1 and a_2 be two distinct finite numbers. If for any $f \in \mathcal{F}$,*

$$\overline{E}_f(a_i) = \overline{E}_{f'}(a_i), \quad i = 1, 2,$$

in the unit disc, then the conclusion of Theorem B holds.

Recently, there exist a lot of studies in using the shared set to obtain the normal family(see. [2, 4, 5]). X.J. Liu obtained a normal function by using the share set $S = \{a_1, a_2, a_3\}$ corresponding Theorem B. Naturally, we ask whether there exists a normal function by using the shared set $S = \{a_1, a_2\}$ corresponding to Corolllary 1.2? In this paper, we study the question and get the following result.

Theorem 1.3 *Let $\mathcal{F}$ be a family of holomorphic functions in the unit disc, a_1 and a_2 be two distinct finite numbers and $a_1 + a_2 \neq 0$. If for any $f \in \mathcal{F}$,*

$$E_f(S) = E_{f'}(S), \quad S = \{a_1, a_2\},$$

in the unit disc, then there exists a positive M, such that for every $f \in \mathcal{F}$, we have

$$(1-|z|^2)f^\sharp(z) < M,$$

where M depends on S.

In the following, we give a example to show the condition $a_1 + a_2 \neq 0$ is necessary.

Example 1.4 ([5]) *Let $S = \{-1, 1\}$. Set $\mathcal{F} = \{f_n(z) : \ n = 2, 3, 4, \ldots\}$, where*

$$f_n(z) = \frac{n+1}{2n}e^{nz} + \frac{n-1}{2n}e^{-nz}, \quad D = \{z : |z| < 1\}.$$

Then, for any $f_n \in \mathcal{F}$, we have

$$n^2[f_n^2(z) - 1] = f_n'^2(z) - 1.$$

Thus f_n and f_n' share S CM, but f_n is not a normal function in D.

From Case 1 in the proof of Theorem 1.3, we can easily get the following corollary.

Corollary 1.5 *Let $\mathcal{F}$ be a family of functions holomorphic in a domain D, let a be a nonzero finite complex numbers. If for all $f \in \mathcal{F}$, f and f' share $S = \{0, \ a\}$ IM, then the conclusion of the theorem 1.3 holds.*

The following example shows that it is necessary that the complex numbers a is finite.

Example 1.6 *Let $S = \{0, \infty\}$. Set $\mathcal{F} = \{e^{nz} : n = 1, 2, \dots\}$ in the unite disc Δ, thus $f_n = e^{nz}$ and $f'_n = ne^{nz}$ share S, but f is not a normal function in Δ.*

Definition 1.7 ([11]) *Given $0 < \alpha < \infty$, if there exists a constant $C_\alpha(f)$ such that*

$$(1 - |z|^2)^\alpha f^\sharp(z) < C_\alpha(f),$$

for each $z \in D$, we say that f is an α-normal function in D.

α-normal functions may be viewed as the generalizations of normal functions. If we denote by N the class of the normal functions in D and denote by N^α the class of the α-normal functions in D, it is obvious that

$$N^{\alpha_1} \subset N \subset N^{\alpha_2}$$

for $0 < \alpha_1 < 1 < \alpha_2 < \infty$. The above inclusion relations are strict(see.[12]). Similarly, we can get the following generalized result.

Theorem 1.8 *Let $\alpha \geq 1$, and let $\mathcal{F}$ be a family of holomorphic functions in the unit disc, a_1 and a_2 be two distinct finite numbers and $a_1 + a_2 \neq 0$. If for any $f \in \mathcal{F}$,*

$$E_f(S) = E_{f'}(S), \quad S = \{a_1, a_2\},$$

in the unit disc, then there exists a positive M, such that for every $f \in \mathcal{F}$, we have

$$(1 - |z|^2)^\alpha f^\sharp(z) < M,$$

where M depends on S.

2 Lemmas

Lemma 2.1 ([9]) *Let $\mathcal{F}$ be a family of functions meromorphic on the unit disc, all of whose zeros have multiplicity at least k, and suppose that there exists $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f \in \mathcal{F}$ and $f(z) = 0$, $f \in \mathcal{F}$. Then if $\mathcal{F}$ is not normal, then there exist, for each $0 \leq \lambda \leq k$,*

(a) a number $0 < r < 1$;

(b) points z_n, $z_n < 1$;

(c) functions $f_n \in \mathcal{F}$, and

(d) positive number $\rho_n \to 0$ such that $\rho_n^{-\lambda} f_n(z_n + a_n\xi) = g_n(\xi) \to g(\xi)$ locally uniformly, where g is a nonconstant meromorphic function on C such that $g^\sharp(\xi) \leq g^\sharp(0) = A + 1$.

The normal lemma is for α-normal functions corresponding to Lemma 2.1.

Lemma 2.2 *Let $\mathcal{F}$ be a family of functions meromorphic on the unit disc, all of whose zeros have multiplicity at least k, and suppose that there exists $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f \in \mathcal{F}$ and $f(z) = 0$, $f \in \mathcal{F}$. Then if $\mathcal{F}$ is not an α-normal function, then there exist, for each $0 \leq \lambda \leq k$ and $1 \leq \alpha < \infty$, there exist a sequence of points $\{z_n\}$ in D and a sequence of positive numbers $\{\rho_n\}$ such that $|z_n| \to 1$, $\rho_n \to 0$, and the sequence of functions*

$$\{g_n(\zeta)\} = \rho_n^{-\lambda} f(z_n + (1 - |z_n|^2)^\alpha \rho_n \zeta)$$

converges spherically and locally uniformly to a non-constant Yosida function in the ζ-plane.

Remark. The case $0 \leq \lambda < k$ is first proved by Chen and Wulan, see [12, 13] for a detail. We can prove the above lemma by the similar method with [13].

3 Proof of Theorem 1.8

Suppose, to the contrary, that we can find $|z_n| < 1$ and $f_n \in \mathcal{F}$ such that

$$g_n(z) = f_n(z_n + (1 - |z_n|^2)^\alpha z) \tag{3.1}$$

satisfy

$$\lim_{n\to\infty} g^\sharp(0) = \lim_{n\to\infty} (1 - |z_n|^2)^\alpha f^\sharp(z_n) = \infty.$$

Hence $\{g_n(z)\}$ is not normal in the unit. By Lemma 2.1, we can find the positive number r, $0 < r < 1$; the complex numbers ζ_n, $|\zeta_n| < 1$; $\rho_n \to 0^+$ and $g_n \in \mathcal{F}$ such that

$$G_n(\zeta) = g_n(\zeta_n + \rho_n\zeta) = f_n(z_n + (1 - |z_n|^2)^\alpha \zeta_n + (1 - |z_n|^2)^\alpha \rho_n \zeta)$$

locally uniformly to a nonconstant entire function $G(\zeta)$ on C.

We know G is a nonconstant entire function. Without loss of generality, we can assume that $G - a_1$ has zeros in C. Let ζ_0 is a zero of $G - a_1$. Consider the family

$$\mathcal{H} = \{H_n(\zeta) : H_n(\zeta) = \frac{G_n(\zeta) - a}{(1 - |z_n|^2)^\alpha \rho_n}\}.$$

We claim $\mathcal{H}$ is not normal at ζ_0. In fact, $G(\zeta_0) = a_1$ and $G(\zeta) \not\equiv a_1$. From (3.1) and Hurwitz's Theorem, there exist ζ_n, $\zeta_n \to \zeta_0$ and $G_n(\zeta_n) = a_1$. Then $H_n(\zeta_n) = 0$. However, there exists a positive number δ such that $\Delta_\delta = \{z \in D : 0 < |\zeta - \zeta_0| < \delta\} \subset D$ and $G(\zeta) \neq a_1$ in Δ_δ. Thus for each $\zeta \in \Delta_\delta$, $G_n(\zeta) \neq a_1$ (for n sufficiently large). Therefore for each $\zeta \in \Delta_\delta$, we have $H(\zeta) = \infty$. Thus we have proved that $\mathcal{H}$ is not normal at ζ_0.

Noting that

$$H_n(\zeta) = 0 \Rightarrow H_n'(\zeta) = a_1 \text{ or } a_2,$$

and using the Lemma 2.1 again we can find $\tau_n \to \tau_0$, $\eta_n \to 0$ and $H_n \in \mathcal{H}$ such that

$$\begin{aligned} F_n(\xi) &= \frac{H_n(\tau_n + \eta_n\xi)}{\eta_n} = \frac{G_n(\tau_n + \rho_n\xi) - a_1}{\eta_n} \\ &= \frac{f_n(z_n + (1 - |z_n|^2)^\alpha \zeta_n + (1 - |z_n|^2)^\alpha \rho_n(\tau_n + \eta_n\xi)) - a_1}{(1 - |z_n|^2)^\alpha \rho_n \eta_n} \end{aligned}$$

locally uniformly convergence to $F(\xi)$ on C, where F is a nonconstant entire function such that $F^\sharp(\xi) \le F^\sharp(0) = M$. In particular $\rho(F) \le 1$.

We claim that

(1) F only has finitely many zeros.

(2) $F(\xi) = 0 \Leftrightarrow F'(\xi) = a_1$ or a_2.

We first prove Claim (1). Suppose ζ_0 is a zero of $G(\zeta) - a_1$ with multiplicity k. If $F(\xi)$ has infinitely many zeros, then there exist $k+1$ distinct points ξ_j $(j = 1, \cdots, k+1)$ satisfying $F(\xi_j) = 0$ $(j = 1, \cdots, k+1)$. Noting that $F(\xi) \not\equiv 0$, by Hurwitz's Theorem, there exists N, if $n > N$, we have $F_n(\xi_{jn}) = 0$ $(j = 1, \cdots, k+1)$ and $G_n(\tau_n + \eta_n\xi_{jn}) - a_1 = 0$. We have

$$\lim_{n\to\infty} \zeta_n + \eta_n\xi_{jn} = \zeta_0, \ (j = 1, \cdots, k+1)$$

then ζ_0 is a zero of $G(\zeta) - a_1$ with multiplicity at least $k+1$, which is a contradiction. Thus we have proved Claim (1).

Next we prove Claim (2). Suppose that $F(\xi_0)=0$, then by Hurwitz's Theorem, there exist ξ_n, $\xi_n\to\xi_0$, such that (for n sufficiently large)

$$F_n(\xi_n)=\frac{f_n(z_n+(1-|z_n|^2)^\alpha\zeta_n+(1-|z_n|^2)^\alpha\rho_n(\tau_n+\eta_n\xi_n))-a_1}{(1-|z_n|^2)^\alpha\rho_n\eta_n}=0.$$

Thus $f_n(z_n+(1-|z_n|^2)^\alpha\zeta_n+(1-|z_n|^2)^\alpha\rho_n(\tau_n+\eta_n\xi_n))=a_1$. By the assumption, we have

$$f_n'(z_n+(1-|z_n|^2)^\alpha\zeta_n+(1-|z_n|^2)^\alpha\rho_n(\tau_n+\eta_n\xi))=a_1 \text{ or } a_2,$$

hence

$$F'(\xi_0)=\lim_{n\to\infty}f_n'(z_n+(1-|z_n|^2)^\alpha\zeta_n+(1-|z_n|^2)^\alpha\rho_n(\tau_n+\eta_n\xi_n))=a_1 \text{ or } a_2.$$

Thus we prove $F(\xi)=0\Rightarrow F'(\xi)=a_1$ or a_2.

In the following, we will prove $F'(\xi)=a_1$ or $a_2\Rightarrow F(\xi)=0$.

Suppose that $F'(\xi_0)=a_1$. Obviously $F'\not\equiv a_1$, for otherwise $F^\sharp(0)\le|F'(0)|=|a_1|<M$, which is a contradiction. Then by Hurwitz's Theorem, there exist ξ_n, $\xi_n\to\xi_0$, such that (for n sufficiently large)

$$F_n'(\xi_n)=f_n'(z_n+(1-|z_n|^2)^\alpha\zeta_n+(1-|z_n|^2)^\alpha\rho_n(\tau_n+\eta_n\xi_n))=a_1.$$

It follows that $F_n(\xi_n)=f_n(z_n+(1-|z_n|^2)^\alpha\zeta_n+(1-|z_n|^2)^\alpha\rho_n(\tau_n+\eta_n\xi_n))=a_1$ or a_2.

If there exists a positive integer N, for each $n>N$, we have

$$f_n(z_n+(1-|z_n|^2)^\alpha\zeta_n+(1-|z_n|^2)^\alpha\rho_n(\tau_n+\eta_n\xi_n))=a_2.$$

Then

$$F(\xi_0)=\lim_{n\to\infty}\frac{f_n(z_n+(1-|z_n|^2)^\alpha\zeta_n+(1-|z_n|^2)^\alpha\rho_n(\tau_n+\eta_n\xi_n))-a_1}{(1-|z_n|^2)^\alpha\rho_n\eta_n}=\infty,$$

it contradicts with $F'(\xi_0)=a_1$. Hence there exists a subsequence of $\{f_n\}$(which, renumbering, we continue to denote by $\{f_n\}$) satisfying that

$$f_n(z_n+(1-|z_n|^2)^\alpha\zeta_n+(1-|z_n|^2)^\alpha\rho_n(\tau_n+\eta_n\xi_n))=a_1.$$

Thus we derive

$$F(\xi_0)=\lim_{n\to\infty}\frac{f_n(z_n+(1-|z_n|^2)^\alpha\zeta_n+(1-|z_n|^2)^\alpha\rho_n(\tau_n+\eta_n\xi_n))-a_1}{(1-|z_n|^2)^\alpha\rho_n\eta_n}=0,$$

which implies $F'=a\Rightarrow F=0$. Similarly, we can get $F'=a_2\Rightarrow F=0$. Hence we have proved claim (2).

Since $\rho(F')=\rho(F)\le1$, then by the Nevanlinna's second fundamental theorem,

$$\begin{aligned}T(r,F')&\le\overline{N}(r,\frac{1}{F'-a_1})+\overline{N}(r,\frac{1}{F'-a_2})+S(r,F')\\&\le\overline{N}(r,\frac{1}{F'-a_1})+\overline{N}(r,\frac{1}{F'-a_2})+O(\log r)\qquad(3.2)\\&\le\overline{N}(r,\frac{1}{F})+O(\log r)\end{aligned}$$

From Claim (1), we get $\overline{N}(r, \frac{1}{F}) = O(\log r)$. Thus $T(r, F') = O(\log r)$, it is clear that F is a polynomial.

In the following, we consider two cases:

Case 1: $a_1 a_2 = 0$. Without loss of generality we assume $a_1 = 0$. We know that F' has zeros, then F has multiple zeros. We assume $\deg(F) = n$, then $T(r, F') = (n-1)\log r$ and $S(r, F') = O(1)$. By (3.2) we get

$$T(r, F') = (n-1)\log r \leq \overline{N}(r, \frac{1}{F}) + O(1) \leq (n-1)\log r$$

Thus we derive that F only has one multiple zeros with multiplicity 2 and F' only has one zero with multiplicity 1, which yields that $n = 2$. Set $F' = B(\xi - \xi_0)$, then $F = (B/2)(\xi - \xi_0)^2$, which contradicts with $F' = a_2 \Rightarrow F = 0$. This completes the proof of Case 1.

Case 2: $a_1 a_2 \neq 0$. We first prove $F = 0 \rightleftharpoons F' = a_1$ or a_2. From $a_1 a_2 \neq 0$, we get $F = 0 \to F' = a_1$ or a_2. Thus we only need to prove $F' = a_1$ or $a_2 \to F = 0$.

Suppose ξ_0 is a zero of $F' - a_1$ with multiplicity m. By Rouché theorem, there exist m sequences $\{\xi_{in}\}(i = 1, 2 \cdots, m)$ on $D_{\delta/2} = \{\xi : |\xi - \xi_0| < \delta/2\}$ such that $F'_n(\xi_{in}) = a_1$. Then

$$f'_n(z_n + (1 - |z_n|^2)^\alpha \zeta_n + (1 - |z_n|^2)^\alpha \rho_n(\tau_n + \eta_n \xi_{in})) = F'_n(\xi_{in}) = a_1 \ (i = 1, 2, \cdots, m).$$

By f and f' share $\{a_1,\ a_2\}$ CM, we get $f' - a_1$ only has simple zeros. That is $\xi_{in} \neq \xi_{jn}(1 \leq i \neq j \leq m)$. We obtain

$$f_n(z_n + (1 - |z_n|^2)^\alpha \zeta_n + (1 - |z_n|^2)^\alpha \rho_n(\tau_n + \eta_n \xi_{in})) = a_1 \text{ or } a_2 \quad (i = 1, 2, \cdots, m).$$

We claim that there exist infinitely many n satisfying

$$f_n\big(z_n + (1 - |z_n|^2)^\alpha \zeta_n + (1 - |z_n|^2)^\alpha \rho_n(\tau_n + \eta_n \xi_{in})\big) = a_1 \quad (i = 1, 2, \cdots, m). \tag{3.3}$$

Otherwise we may assume that for all n, there exist $j \in (1, \ldots, m)$ satisfying

$$f_n(z_n + (1 - |z_n|^2)^\alpha \zeta_n + (1 - |z_n|^2)^\alpha \rho_n(\tau_n + \eta_n \xi_{in})) = a_2.$$

We take a fixed number $l \in (1, ..., m)$ satisfying (for infinitely many n)

$$f_n(z_n + (1 - |z_n|^2)^\alpha \zeta_n + (1 - |z_n|^2)^\alpha \rho_n(\tau_n + \eta_n \xi_{in})) = a_2.$$

Hence

$$\begin{aligned} F(\xi_0) &= \lim_{n\to\infty} \frac{f'_n(z_n + (1 - |z_n|^2)^\alpha \zeta_n + (1 - |z_n|^2)^\alpha \rho_n(\tau_n + \eta_n \xi_{in})) - a_1}{(1 - |z_n|^2)^\alpha \rho_n \eta_n} \\ &= \lim_{n\to\infty} \frac{a_2 - a_1}{(1 - |z_n|^2)^\alpha \rho_n \eta_n} = \infty, \end{aligned}$$

which contradicts with $F'(\xi_0) = a_1$. This proves (3.3). Therefore,

$$F_n(\xi_{in}) = 0, \quad (i = 1, 2, \cdots, m)$$

and $\xi_{in} \neq \xi_{jn}$ $(1 \leq i \neq j \leq m)$. As $n \to \infty$, we get ξ_0 is a zero of F with multiplicity at least m. This proves $F' = a_1 \to F = 0$. Similarly we can get $F' = a_2 \to F = 0$. Thus we have proved

$$F = 0 \rightleftharpoons F' = a_1 \text{ or } a_2.$$

From this we know $F' - a_1$ and $F' - a_2$ only have simple zeros. Suppose that $\deg(F) = n$, then $n = 2(n-1)$ and $n = 2$. Set $F = A(\xi - \xi_1)(\xi - \xi_2)$, then $F' = A(2\xi - \xi_1 - \xi_2)$.

Without loss of generality, we assume that $F'(\xi_1) = a_1$ and $F'(\xi_2) = a_2$, we get $a_1 + a_2 = 0$. It is a contradiction.

Thus we complete the proof of Theorem 1.3.

Bibliography

1. J. Clunie, W.K. Hayman. The spherical derivative of integral and meromorphic functions, Comment.Math.Helv. 40(1966), 117-148.

2. M.L. Fang, L. Zalcman. Normality and shared sets, J. Aust. Math. Soc., 86(2009), 339-354.

3. G. Gundersen. Meromorphic functions that share finite values with their derivative, J. of Math. Anal and Appl. 75(1980), 441-446.

4. X.J. Liu, X.C. Pang. Shared values and normal function, Acta Mathematica Sinca, Chinese Series, 50(2007), 409-412.

5. F.Lü, J.F. Xu. Sharing set and normal families of entire functions and their derivatives. Houston J. Math. 34(2008), 1213–1223.

6. P. Lappan. A criterion for a meromorphic function to be normal, Comm. Math. Helv. 49(1974), 492–495.

7. O. Lehto, K.I. Virtanen. Boundary behaviour and normal meromorphic functions, Acta Math. 97 (1957), 47–65.

8. X.C. Pang. Normal family and normal function of meromorphic function, Chinese. Ann. Math., 21A, 5(2000), 601-604.

9. X.C. Pang, L.Zalcman. Normal families and shared values, Bull. London Math. Soc. 32(2000), 325-331.

10. J. Schiff. Normal families, Springer (1993).

11. H. Wulan. On some classes of meromorphic functions, Ann. Acad. Sci. Fenn. Mathematica, Dissertations, p. 116.

12. Y. Xu. The α-normal functions., Computers and Math. Appl. 44(2002), 357-363.

13. Y. Xu. Normal functions and α-normal functions., Acta Mathematica Sinica, English Series, 16(2000), 399-404.

14. W. Schwick. Sharing values and normality, Arch math.(Basel) 59(1992), 50-54.

15. H.X. Yi, C.C. Yang. The Uniqueness Theory of Meromorphic Functions, New York, Kluwer Academic Publishers, 2003.

РАЗДЕЛЕННОЕ МНОЖЕСТВО И НОРМАЛЬНАЯ ФУНКЦИЯ ГОЛОМОРФНЫХ ОТОБРАЖЕНИЙ

Жан Фенг Ксю

Вууй Университет,

Цзянмэнь, Гуангдонг 529020, Китай, e-mail: lvfeng@mail.sdu.edu.cn

Аннотация. В работе идея разделенного множества применяется к описанию нормальных функций для семейства мероморфных функций в единичном круге.

Ключевые слова: целая функция, единственность, теория Неванлинна.

2000 MSC 30D35

UNIQUENESS OF MEROMORPHIC FUNCTIONS THAT SHARE THREE VALUES

Jian-Ping Wang, Xiao-Ling Wang, Ling-Di Huang
Shaoxing College of Arts and Sciences,
Shaoxing, Zhejiang 312000, People's Republic of China, e-mail: jpwang604@hotmail.com

Abstract. In this paper, we study the uniqueness of meromorphic functions that share three values or three small functions with the same multiplicities and prove some results on this topic given by G. Brosch, X. H. Hua and M. L. Fang, etc.

Keywords: uniqueness, meromorphic function, value sharing.

1. Introduction and Results

It is assumed that the reader is familiar with the usual notations and the fundamental results of R. Nevanlinna theory of meromorphic function as found in [5].

Let $f,\ g$ be nonconstant meromorphic functions. We say that a meromorphic function $a(z)(\not\equiv\infty)$ is a small function of f if $T(r,a)=S(r,f)$. If $N(r,1/(f-a))=S(r,f)$, then we say that a is an exceptional function of f. Moreover, we denote by $N(r,f=a=g)$ the counting function of those common zeros of $f-a$ and $g-a$, where z_0 is counted $\min\{p,\ q\}$ times if z_0 is a common zero of $f-a$ and $g-a$ with multiplicity p and q respectively; as usual, by $\overline{N}(r,f=a=g)$ the corresponding reduced counting function; and by $N_E(r,f=a=g)$ the counting function which "counts" only those common zeros of $f-a$ and $g-a$ with the same multiplicity in $N(r,f=a=g)$. These notations will be used throughout the paper.

Let $f,\ g$ be two nonconstant meromorphic functions, and let a be a small function of f and g or a be a constant. We say that f and g share a CM if $f-a$ and $g-a$ have the same zeros with the same multiplicity; if we ignore the multiplicity, then we say that f and g share a IM. For the statement of our results, we may need a slightly generalization of the definitions of CM and IM (see [6],[8]).

In 1997, Hua and Fang proved the following result.

Theorem A[6]. *Let f and g be two nonconstant meromorphic functions, and let $a_j(z)$ $(j=1,\cdots,4)$ be distinct small functions of f and g. If f and g share $a_j(z)$ $(j=1,2,3)$ CM, and share $a_4(z)$ IM. Then f and g satisfy one of the following cases.*

(i) $f\equiv g$, (ii) $F\equiv -G$ with $a(z)\equiv -1$, (iii) $F+G\equiv 2$ with $a(z)\equiv 2$,
(iv) $(F-1/2)(G-1/2)\equiv 1/4$ with $a(z)\equiv 1/2$, (v) $F\cdot G\equiv 1$ with $a(z)\equiv -1$,
(vi) $(F-1)(G-1)\equiv 1$ with $a(z)\equiv 2$, (vii) $F+G\equiv 1$ with $a(z)\equiv 1/2$,

where $F\equiv \frac{f-a_1}{f-a_3}\frac{a_2-a_3}{a_2-a_1}$, $G\equiv \frac{g-a_1}{g-a_3}\frac{a_2-a_3}{a_2-a_1}$, and $a(z)\equiv \frac{a_4-a_1}{a_4-a_3}\frac{a_2-a_3}{a_2-a_1}$.

Remark 1. From the proof of Lemma 6 and Lemma 7 in [6], it is easy to see that the conclusion is still true if we replace IM with "IM" in Theorem A.

Project supported by the National Natural Science Foundation(Grant No.10771121) of China, and the Natural Science Foundation(Grant No.Y6090641) of Zhejiang Province.

For the meromorphic functions that share three values, G. Brosch proved

Theorem B(see [1] or [11]). *Let two meromorphic functions f and g share 0, 1, ∞ CM. If there exists a finite value $a(\neq 0, 1)$ such that $g(z) = a$ whenever $f(z) = a$. Then f is a Möbius transformation of g.*

In 2008, two of the present authors proved a result on this topic.

Theorem C(see [15, Theorem 2]). *Let two nonconstant meromorphic functions f and g share 0,1, ∞ CM. If there exists a small entire function $a(z)(\not\equiv 0, 1, \infty)$ of f and g such that $g(z) - a(z) = 0$ whenever $f(z) \overset{(p)}{=} a(z)$ for $p = 1, 2$. Then f and g must satisfy one of the following ten cases.*

(i) $f \equiv g$, (ii) $f \equiv ag$, where $a(z)(\not\equiv -1), 1$ are exceptional functions of f,
(iii) $f - 1 \equiv (1 - a)(g - 1)$, where $a(z)(\not\equiv 2), 0$ are exceptional functions of f,
(iv) $(f - a)(g - 1 + a) \equiv a(1 - a)$, where $a(z)(\not\equiv \frac{1}{2}), \infty$ are exceptional functions of f, (v)$f \equiv -g$ with $a(z) \equiv -1$, (vi) $f + g \equiv 2$ with $a(z) \equiv 2$,
(vii) $(f - \frac{1}{2})(g - \frac{1}{2}) \equiv \frac{1}{4}$ with $a(z) \equiv \frac{1}{2}$, (viii) $f \cdot g \equiv 1$ with $a(z) \equiv -1$,
(ix) $(f - 1)(g - 1) \equiv 1$ with $a(z) \equiv 2$, (x) $f + g \equiv 1$ with $a(z) \equiv \frac{1}{2}$.

The main purpose of this paper is further to study the uniqueness of meromorphic functions that share three values or three small functions with the same multiplicities, and to prove the following three results.

Theorem 1. *Let two nonconstant meromorphic functions f and g share 0,1, ∞ CM. If there exists a small function $a(z)(\not\equiv 0, 1, \infty)$ of f and g such that $N(r, f = a = g) \neq S(r, f)$. Then f and g satisfy one of the following five cases.*

(i) $f \equiv g$, (ii) $f \cdot g \equiv 1$ with $a(z) \equiv -1$, (iii) $f + g \equiv 1$ with $a(z) \equiv \frac{1}{2}$,
(iv) $(f - 1)(g - 1) \equiv 1$ with $a(z) \equiv 2$,
(v)$f(z) = \frac{e^{\int a(z)\gamma'(z)dz} - 1}{e^{\gamma(z)} - 1}$, $g(z) = \frac{e^{-\int a(z)\gamma'(z)dz} - 1}{e^{-\gamma(z)} - 1}$,
where $\gamma(z)$ is a nonconstant entire function, and $a(z) \not\equiv -1, 1/2, 2$.

Let f be a meromorphic function, let a be a small function of f or be a constant, and let p be a positive integer. We denote by $f(z_0) \overset{(p)}{=} a$ that z_0 is a zero of $f - a$ with multiplicity p. By the above Theorem 1, we can prove the following result which generalize the small function $a(z)$ in Theorem C from entire to meromorphic, and is also a great improvement of Theorem B. In order to avoid needless duplication, we shall omit the details of the proof of the following Theorem 2 in this paper.

Theorem 2. *Let two nonconstant meromorphic functions f and g share 0,1, ∞ CM. If there exists a small function $a(z)(\not\equiv 0, 1, \infty)$ of f and g such that $g(z) - a(z) = 0$ whenever $f(z) \overset{(p)}{=} a(z)$ for $p = 1, 2$. Then the conclusion of Theorem C still holds.*

From Theorem 2, we can immediately obtain the following result which improves and generalizes Theorem A.

Theorem 3. *Let F and G be nonconstant meromorphic functions, and let $a_j(z)(j = 1,2,3,4)$ be distinct small functions of F and G. If F and G share $a_j(z)(j = 1,2,3)$ CM, and if $G(z) = a_4(z)$ whenever $F(z) = a_4(z)$. Then f and g satisfy the conclusion of Theorem C, where $f \equiv \frac{F-a_1}{F-a_3}\frac{a_2-a_3}{a_2-a_1}$, $g \equiv \frac{G-a_1}{G-a_3}\frac{a_2-a_3}{a_2-a_1}$ and $a \equiv \frac{a_4-a_1}{a_4-a_3}\frac{a_2-a_3}{a_2-a_1}$.*

2. Lemmas

Lemma 1 (see [16]). *Suppose that $f_1, f_2, \cdots, f_n$ $(n \geq 3)$ are meromorphic functions which are not constants except for f_n. Furthermore, let $\sum\limits_{j=1}^{n} f_j(z) \equiv 1$. If $f_n(z) \not\equiv 0$, and*

$$\sum_{j=1}^{n} N(r, 1/f_j) + (n-1)\sum_{j=1}^{n} \overline{N}(r, f_j) < (\lambda + o(1))T(r, f_k),$$

where $r \in I, k = 1, 2, \cdots, n-1$ and $\lambda < 1$, then $f_n(z) \equiv 1$.

Lemma 2(see [16]). *Let $f_1,\ f_2$ be nonconstant meromorphic functions and c_1, c_2, c_3 be non-zero constants. If $c_1 f_1 + c_2 f_2 \equiv c_3$, then*

$$T(r, f_1) < \overline{N}(r, 1/f_1) + \overline{N}(r, 1/f_2) + \overline{N}(r, f_1) + S(r, f_1).$$

Lemma 3(see [6, Lemma 5]). *Let f and g be two nonconstant meromorphic functions that share $0, 1, \infty$ CM. If $f \not\equiv g$, then for any small function $a(z)(\not\equiv 0, 1, \infty)$ of f and g, we have*

$$N_{(3}\left(r, \frac{1}{f-a}\right) + N_{(3}\left(r, \frac{1}{g-a}\right) = S(r, f).$$

3. The Proof of Theorem 1

We suppose first that $f \not\equiv g$. Since f and g share $0, 1, \infty$ CM, by the second fundamental theorem due to R. Nevanlinna, we have

$$\begin{aligned}(1 + o(1))T(r, f) &\leq N(r, f) + N(r, \tfrac{1}{f}) + N(r, \tfrac{1}{f-1}) \\ &\leq N(r, g) + N(r, \tfrac{1}{g}) + N(r, \tfrac{1}{g-1}) \leq (3 + o(1))T(r, g). \qquad (3.1)\end{aligned}$$

Similarly, we obtain

$$(1 + o(1))T(r, g) \leq (3 + o(1))T(r, f). \tag{3.2}$$

From (3.1) and (3.2), it follows that

$$S(r, f) = S(r, g). \tag{3.3}$$

Set

$$\varphi := \frac{f'(f-a)}{f(f-1)} - \frac{g'(g-a)}{g(g-1)}. \tag{3.4}$$

If $\varphi \not\equiv 0$, then from (3.3), (3.4), the fundamental estimate of the logarithmic derivative, and the hypothesis that f and g share $0, 1, \infty$ CM, we have

$$T(r, \varphi) = S(r, f) + S(r, g) = S(r, f). \tag{3.5}$$

Since f and g share $0, 1, \infty$ CM, thus by (3.4) and (3.5) we deduce that

$$N(r, f = a = g) \leq N(r, 1/\varphi) + S(r, f) \leq T(r, \varphi) + S(r, f) = S(r, f),$$

which contradicts the hypothesis of Theorem 1. Hence, we have $\varphi \equiv 0$, namely

$$\frac{f'(f-a)}{f(f-1)} \equiv \frac{g'(g-a)}{g(g-1)}. \tag{3.6}$$

Noting that f and g share $0, 1, \infty$ CM, thus there exist two entire functions α and β such that

$$\frac{f}{g} = e^{\alpha}, \qquad \frac{f-1}{g-1} = e^{\beta}. \tag{3.7}$$

Since $f \not\equiv g$, by (3.7) we can deduce that $e^{\alpha} \not\equiv 1$, $e^{\beta} \not\equiv 1$ and $e^{\beta-\alpha} \not\equiv 1$. Set $\gamma := \beta - \alpha$, then from (3.7) we have

$$f = \frac{e^{\beta}-1}{e^{\gamma}-1}, \qquad g = \frac{e^{-\beta}-1}{e^{-\gamma}-1}. \tag{3.8}$$

Rewriting (3.6) as

$$(1-a)\left(\frac{f'}{f-1} - \frac{g'}{g-1}\right) \equiv a\left(\frac{g'}{g} - \frac{f'}{f}\right). \tag{3.9}$$

By (3.7) and the fact that $\alpha = \beta - \gamma$, we obtain

$$\frac{f}{g} = e^{\beta-\gamma}, \qquad \frac{f-1}{g-1} = e^{\beta}, \tag{3.10}$$

from (3.10), it follows that

$$\frac{f'}{f} - \frac{g'}{g} = \beta' - \gamma', \quad \frac{f'}{f-1} - \frac{g'}{g-1} = \beta'. \tag{3.11}$$

Substitution (3.11) into (3.9) gives

$$\beta' \equiv a\gamma'. \tag{3.12}$$

From (3.8) and (3.12), we have

$$f = \frac{e^{\int a\gamma'}-1}{e^{\gamma}-1}, \qquad g = \frac{e^{-\int a\gamma'}-1}{e^{-\gamma}-1}. \tag{3.13}$$

We now claim that $[a(z)+1][a(z)-\frac{1}{2}][a(z)-2] \equiv 0$ if and only if f and g satisfy one of the cases (ii)-(iv) of the conclusion of Theorem 1, and thus f is a Möbius transformation of g.

In fact, if $a(z) \equiv \frac{1}{2}$, then from (3.12) we have $\gamma \equiv 2\beta + c$, where c is a constant. Thus, by (3.7) and the fact that $\alpha = \beta - \gamma$, it follows that

$$\frac{g}{f} \equiv e^{\gamma-\beta} \equiv e^{\beta+c} \equiv e^{c}\frac{f-1}{g-1}. \tag{3.14}$$

Noting that $N(r,f=a=g)\neq S(r,f)$, we can deduce that there exists a point z_0 such that $f(z_0)=g(z_0)=a(z_0)$ $(\neq 0,1,\infty)$, which and (3.14) imply that $e^c=1$, and thus we obtain from (3.14) that $(g-f)(g+f-1)\equiv 0$, that is $f+g\equiv 1$. Similarly, if $a(z)\equiv -1$ or $a(z)\equiv 2$, then from (3.7), (3.12), the fact $\alpha=\beta-\gamma$, and the hypothesis of Theorem 1, we can also obtain that $f\cdot g\equiv 1$ or $(f-1)(g-1)\equiv 1$, respectively.

On the other hand, suppose that there exist four finite complex numbers c_j $(j=1,2,3,4)$ such that $f=\frac{c_1g+c_2}{c_3g+c_4}$, where $c_1c_4\neq c_2c_3$. By this and (3.13) we get

$$\begin{aligned} 2c_3+c_4-2c_2-c_1 &= c_1e^{\gamma-\int a\gamma'}+(c_3-c_1)e^{-\int a\gamma'}+(c_3+c_4)e^{\int a\gamma'} \\ &\quad -c_4e^{-\gamma+\int a\gamma'}-(c_1+c_2)e^{\gamma}+(c_4-c_2)e^{-\gamma}. \end{aligned} \tag{3.15}$$

We note first that γ is not a constant. Otherwise, from (3.12) we know that β is also a constant, and thus by (3.8) we can deduce that f is a constant, a contradiction. So from this and the fact that $a(z)\not\equiv 0,1$, we can also derive that both $\gamma-\int a\gamma'$ and $\int a\gamma'$ are not constants. In the sequel, by repeatedly applying Lemma 1 to equality (3.15) and its modified forms, and noting the fact that $c_1c_4\neq c_2c_3$, and that $a(z)\not\equiv 0,1$, we can prove that one of the following cases holds.

(a) $\gamma-2\int a\gamma'\equiv$ constant, that is $a(z)\equiv \frac{1}{2}$,
(b) $2\gamma-\int a\gamma'\equiv$constant, that is $a(z)\equiv 2$, and
(c) $\gamma+\int a\gamma'\equiv$constant, that is $a(z)\equiv -1$.

For this purpose, we shall divide our argument into two cases.

Case 1. $A:=2c_3+c_4-2c_2-c_1=0$.
From (3.15) we have

$$c_1e^{\gamma-\int a\gamma'}+(c_3-c_1)e^{-\int a\gamma'}+(c_3+c_4)e^{\int a\gamma'}-c_4e^{-\gamma+\int a\gamma'}-(c_1+c_2)e^{\gamma}+(c_4-c_2)e^{-\gamma}\equiv 0. \tag{3.16}$$

We now need to consider the following seven subcases.
Subcase 1.1. $c_1c_4(c_3-c_1)(c_3+c_4)(c_1+c_2)(c_4-c_2)\neq 0$. Rewrite (3.16) as

$$\frac{c_1}{c_2-c_4}e^{2\gamma-\int a\gamma'}+\frac{c_3-c_1}{c_2-c_4}e^{\gamma-\int a\gamma'}+\frac{c_3+c_4}{c_2-c_4}e^{\gamma+\int a\gamma'}-\frac{c_4}{c_2-c_4}e^{\int a\gamma'}-\frac{c_1+c_2}{c_2-c_4}e^{2\gamma}\equiv 1. \tag{3.17}$$

Suppose that $\gamma+\int a\gamma'\not\equiv$ *constant*. Noting the fact that $\gamma-\int a\gamma'$, $\int a\gamma'$, and γ are all not constant, so we can get by applying Lemma 1 to (3.17) that $\frac{c_1}{c_2-c_4}e^{2\gamma-\int a\gamma'}\equiv 1$, and thus from (3.17) it follows that

$$\frac{c_3-c_1}{c_1+c_2}e^{-\gamma-\int a\gamma'}+\frac{c_3+c_4}{c_1+c_2}e^{-\gamma+\int a\gamma'}-\frac{c_4}{c_1+c_2}e^{-2\gamma+\int a\gamma'}\equiv 1. \tag{3.18}$$

By Lemma 1 and (3.18), we get $-\frac{c_4}{c_1+c_2}e^{-2\gamma+\int a\gamma'}\equiv 1$. From this and (3.18) we get $\int a\gamma'\equiv$ *constant*, a contradiction.

Suppose that $\gamma+\int a\gamma'\equiv$ *constant*. Then we must have $2\gamma-\int a\gamma'\not\equiv$ *constant*. Otherwise, we shall find that γ is a constant, which is impossible. Thus, from (3.17) and Lemma 1 we get $\frac{c_3+c_4}{c_2-c_4}e^{\gamma+\int a\gamma'}\equiv 1$, and thus again from (3.17) and Lemma 1 we have

$$\frac{c_1}{c_1+c_2}e^{-\int a\gamma'}+\frac{c_3-c_1}{c_1+c_2}e^{-\gamma-\int a\gamma'}-\frac{c_4}{c_1+c_2}e^{-2\gamma+\int a\gamma'}\equiv 1. \tag{3.19}$$

Noting the assumption $\gamma + \int a\gamma' \equiv constant$, so we must have $-2\gamma + \int a\gamma' \not\equiv constant$. By applying Lemma 1 to (3.19), we deduce that $\gamma - \int a\gamma' \equiv constant$, this is also a contradiction. Therefore, the subcase 1.1 can not occur.

Next, we can use the similar method to deal with the following six subcases: $c_1 = 0$; $c_4 = 0$ but $c_1 \neq 0$; $c_3 - c_1 = 0$, but $c_1c_4 \neq 0$; $c_3 + c_4 = 0$, but $c_1c_4(c_3 - c_1) \neq 0$; $c_1 + c_2 = 0$, but $c_1c_4(c_3 - c_1)(c_3 + c_4) \neq 0$; $c_2 - c_4 = 0$. For the sake of simplicity, we omit the details.

Case 2. $A := 2c_3 + c_4 - 2c_2 - c_1 \neq 0$.

In fact, we shall verify that the case 2 can not occur by dividing it into five subcases. In case 2, from (3.15) we have

$$\frac{c_1}{A}e^{\gamma - \int a\gamma'} + \frac{c_3 - c_1}{A}e^{-\int a\gamma'} + \frac{c_3 + c_4}{A}e^{\int a\gamma'} - \frac{c_4}{A}e^{-\gamma + \int a\gamma'} - \frac{c_1 + c_2}{A}e^{\gamma} + \frac{c_4 - c_2}{A}e^{-\gamma} \equiv 1. \quad (3.20)$$

If $c_1c_4(c_3 - c_1)(c_3 + c_4)(c_1 + c_2)(c_4 - c_2) \neq 0$, then by (3.20) and Lemma 1, we can get a contradiction by noting that $\gamma, \int a\gamma'$ and $\gamma - \int a\gamma'$ are all not constants. So we know that at least one of the six numbers is zero.

Next, we consider the following five subcases.

Subcase 2.1. $c_1 = 0$. In this subcase, we have $c_2c_3 \neq 0$. By (3.20) we obtain

$$\frac{c_3}{A}e^{-\int a\gamma'} + \frac{c_3 + c_4}{A}e^{\int a\gamma'} - \frac{c_4}{A}e^{-\gamma + \int a\gamma'} - \frac{c_2}{A}e^{\gamma} + \frac{c_4 - c_2}{A}e^{-\gamma} \equiv 1. \quad (3.21)$$

If $c_3 + c_4 = 0$, then $c_4 = -c_3 \neq 0$. So, from (3.21) and Lemma 1 we get $c_4 - c_2 = 0$, and thus a contradiction.

If $c_3 + c_4 \neq 0$, then we must have $c_4 \neq 0$. Otherwise, by applying Lemma 1 to (3.21), we can get a contradiction. Now again by (3.21) and Lemma 1 we get $c_4 - c_2 = 0$, and thus a contradiction . Thus we have $c_1 \neq 0$.

We can easily dealt with the other four subcases $c_4 = 0$; $c_3 - c_1 = 0$; $c_3 + c_4 = 0$; $c_1 + c_2 = 0$ by the similar method.

In the above five subcases, we have shown that $c_1c_4(c_3 - c_1)(c_3 + c_4)(c_1 + c_2) \neq 0$. Therefore, we can always obtain a contradiction by using Lemma 1 to (3.20) whether $c_4 - c_2 = 0$ holds or not. The proof of Theorem 1 is completed.

Bibliography

1. G. Brosch. Eindeutigkeitssätze für meromorphe funktionen, Thesis, Technical University of Aachen, 1989.

2. G.G. Gundersen. Meromorphic functions that share three values IM and a fourth values CM, Complex Variables Theory Appl., 20(1992), 99-106.

3. G.G. Gundersen. Meromorphic functions that share three or four values, J. London Math. Soc., 20(1979), 457-466.

4. G.G. Gundersen. Meromorphic functions that share four values, Trans. Amer. Math. Soc., 277(1983), 545-567.

5. W.K. Hayman. Meromorphic Functions, Oxford: Clarendon, 1964.

6. Xin-Hou Hua, Ming-Liang Fang. Meromorphic functions sharing four small functions, Indian J. pure appl. Math., 28(1997), 797-811.

7. K. Ishizaki. Meromorphic functions sharing small functions, Arch. Math., 77(2001), 273-277.

8. P. Li. Meromorphic functions that share four small functioms, J. Math. Anal. Appl., 263(2001), 316-326.

9. Yu-Hua Li, Jian-Yong Qiao. The uniqueness of meromorphic functions concerning small functions, Science in China(Series A), 43(6)(2000), 581-590.

10. Xiao-Min Li, Hong-Xun Yi. Meromorphic functions sharing three values. J. Math. Soc. Japan, 56(1)(2004), 147–167.

11. E. Mues. Shared value problems for meromorphic functions, Joensuu: University of Joensuu, Publ. Sci. Univ. Joensuu, 35(1995), 17-43, (ISBN 951-708-379-3/pbk).

12. R. Nevanlinna. Le th*é*or*è*me de Picard-Borel et la th*é*orie des fonctions meromorphes, Gauthier-Villars, Paris, 1929.

13. H. Ueda. Meromorphic functions f and g that share two values CM and two other values in the sense of $E_k(\beta, f) = E_k(\beta, g)$, Kodai Math. J., 21(1998), 273-284.

14. Jian-Ping Wang. Meromorphic functions sharing four values, Indian J. pure Appl. Math., 32(1)(2001), 37-46.

15. Jian-Ping Wang. Ling-Di Huang, Meromorphic functions sharing three values CM, Kodai Math. J., 31(1)(2008), 150-162.

16. C.C. Yang, H.X. Yi. Uniqueness Theory of Meromorphic Functions, Dordrecht/Beijing/ New York: Kluwer Academic Publishers and Science Press, 2003.

17. H.X. Yi. On one problem of uniqueness of meromorphic functions concerning small functions, Proc. Amer. Math. Soc., 130(2001), 1689-1697.

18. Q.C. Zhang. Meromorphic functions sharing three valus, Indian J. pure appl. Math., 30(1999), 667-682.

ЕДИНСТВЕННОСТЬ МЕРОМОРФНЫХ ФУНКЦИЙ С ТРЕМЯ РАЗДЕЛЕННЫМИ ЗНАЧЕНИЯМИ

Янг-Пинг Ванг, Ксяо Линг Ванг, Линг Ди Хуанг
Шаосинский колледж искусства и науки,
Чжэцзян, Шаосин, 312000, Китай e-mail: jpwang604@hotmail.com

Аннотация. В работе изучаются единственность мероморфных функций с тремя разделяющимися значениями или малыми функциями той же кратности.

Ключевые слова: единственность, мероморфная функция, разделяющиеся значения.

УДК 517.9

THE FIRST ORDER SYSTEM EQUATIONS OF A PRINCIPAL TYPE ON THE PLANE

N.A. Zhura

Lebedev Physical Institute of the Russian Academy of Sciences,
Leninskij pr., 53, LPI, Moscow, 119991, Russian Federation, e-mail: nikzhura@gmail.com

Abstract. Boundary value problems for the system equations of a principal type with constant coefficients on the plane are studied. The half-infinite domains with noncharacteristic boundary and finite domains with such property are considered. The representation solutions of this systems through solutions of canonical elliptic and hyperbolic systems is obtained. Also the index formula for associated problems in Holder weighted classes is founded.

Keywords: principal type equations, noncharacteristic boundary, index formula, function theoretical approach, canonical systems of first order.

1 Integral representation

On the (x_1, x_2) - plane $\mathbb{R}^2$ we consider a system of linear partial differential equations

$$\frac{\partial u}{\partial x_2} - a\frac{\partial u}{\partial x_1} = 0, \tag{1}$$

where $u(x)$ is an unknown l - vector-valued function and $a \in \mathbb{R}^{l\times l}$ is a constant matrix. The system (1) is said to be of a composite type (principal one [1, 2]) if it's characteristic equation

$$\det(a - \nu) = 0 \tag{2}$$

has $s_1 \geq 1$ complex roots with the positive imaginary part and $s_2 \geq 1$ real roots, $2s_1 + s_2 = l$.

Let $b_1 \in C^{s_1\times l}$, $b_2 \in \mathbb{R}^{s_2\times l}$ be constant matrices such that nonsingular matrix $b = (b_1|\overline{b_1}|b_2)$ reduces a to the Jordan normal form

$$b^{-1}ab = \mathrm{diag}\,(J_1, \overline{J_1}, J_2), \tag{3}$$

where the block matrixes $J_k \in \mathbb{C}^{s_k\times s_k}$, $k = 1, 2$, are composed from Jordan cells. Here J_1 has complex eigenvalues with positive imaginary part and $J_2 \in \mathbb{R}^{s_2\times s_2}$ has only real eigenvalues. Let $k_2 \leq s_2$ denote the maximum of orders of Jordan cells composing J_2.

It is valid the following representation theorem [3].

Theorem 1. *Any regular solution u of the system* (1) *can be represented in the form*

$$u = 2\mathrm{Re}\, b_1\Phi + b_2\Psi, \tag{4}$$

where Φ is a regular solution of the canonical elliptic system

$$\frac{\partial \Phi}{\partial x_2} = J_1\frac{\partial \Phi}{\partial x_1}, \tag{5}$$

The work was supported in part by the National Natural Science Foundation of China (NSFC) in the framework of the bilateral project "Complex Analysis and its applications"(project No. 08-01-92208-GFEN)

and Ψ *is a regular solution of the canonical hyperbolic system*

$$\frac{\partial \Psi}{\partial x_2} = J_2 \frac{\partial \Psi}{\partial x_1}. \tag{6}$$

Solutions of the system (5) are said to be a J_1 – analytical functions. It is known [4] that a general solution of this systems can be represented in the form

$$\Phi(x) = \left[\exp(x_2 J_{1,0}) \frac{\partial}{\partial x_1} \right] \phi(x_1 + \nu x_2),$$

where $J_1 = \nu + J_{10}$ is decomposition of the matrix J_1 into diagonal ν and nilpotent parts. Here the s_1- vector $\phi(x_1 + \nu x_2)$ consists from components $\phi_j(x_1 + \nu_j x_2)$, $1 \leq j \leq s_1$, where the functions $\phi_j(z)$ are analytic in the corresponding domain of the complex plane. The similar representation

$$\Psi(x) = \left[\exp(x_2 J_{2,0}) \frac{\partial}{\partial x_1} \right] \psi(x), \tag{7}$$

there exists for a s_2-vector-valued function $\psi = (\psi_j,\, 1 \leq j \leq s_2)$, where $J_2 = \nu + J_{2,0}$ and $\psi_j(x) = \tilde{\psi}_k(x_1 + \nu_j x_2)$. Note that ψ_j satisfies the hyperbolic equation

$$\frac{\partial \psi_j}{\partial x_2} = \nu_j \frac{\partial \psi_j}{\partial x_1}, \quad \nu_j \in \mathbb{R}.$$

2 Fredholm solvability in the half-infinite domain

Let $C^\mu(\overline{D})$ be the space of functions satisfying the Holder condition on the the closed domain $\overline{D}$ with exponent $0 < \mu \leq 1$ (and bounded if $\overline{D}$ is infinite). The space $C^{\mu,n}(\overline{D})$ consists of the functions with partial derivatives in $C^{\mu,n-1}$, $n \geq 1$, $(C^{\mu,0} = C^\mu)$. These spaces are Banach with respect to the corresponding norm. It is convenient to write $C^{\mu+0,n}$ for the class $\cup_{\varepsilon>0} C^{\mu+\varepsilon,n}$.

If the domain D is infinite we also use the space $C^{\mu,n}(\hat{D})$ for the set $\hat{D} = \overline{D} \cup \{\infty\}$ considered as the compact on the Riemann sphere $\hat{\mathbb{C}}$. These definitions also applies to the classes $C^{\mu,n}$ on curves $\Gamma \subseteq \mathbb{C}$.

Let D be a half-infinite domain on the complex plane i.e. it is a simple connected domain with smooth boundary Γ on the Riemann sphere. So the curve Γ permits a smooth parametrization $z = \gamma(t)$, $t \in \mathbb{R}$, where

$$\gamma'(t) \in C^{\mu,k_2}(\hat{\mathbb{R}}). \tag{8}$$

We consider a boundary value problem

$$Cu = f \text{ on } \Gamma, \tag{9}$$

for the system (1) where C is a $(s_1 + s_2) \times l$ matrix-valued function, and f is a $(s_1 + s_2)$ vector-valued function on $\Gamma = \partial D$. This problem is considered in the class $C^{\mu,1}(\overline{D})$ of solutions (1) such that the functions Φ and Ψ belong to this class in the representation (4). More exactly we say that the vector-valued function Ψ defined by (7) belongs to the class $C^{\mu,1}$ if the components of $\tilde{\psi}$ belong to the class $C^{s_2+1-j,\mu}$, $j = 1, \ldots, s_2$, as functions of one variable. For brevity it is assumed here that J_2 consists from one Jordan cell, in the general case this definition is regarded with respect to each Jordan's block of J_2. In what follows it is assumed that the characteristics $x_1 + \nu_j x_2 = \text{const}$ of the system (6) don't tangent of the curve Γ, i.e.

$$\mathrm{Re}\gamma'(t) + \nu_j \mathrm{Im}\gamma'(t) \neq 0, \quad t \in \hat{\mathbb{R}},\ 1 \leq j \leq s_2. \tag{10}$$

Moreover it is assumed that Γ coincide with a straight line in a neighborhood of ∞.

It is assumed also, that

$$C, f \in C^{\mu,k_2}(\hat{\Gamma}) \tag{11}$$

and

$$|(b^{-1}u)_1| \leq \text{const}(|x|)^{-1} \tag{12}$$

as $|x| \to \infty$, where by $(b^{-1}u)_1$ we denote the first s_1 elements of the vector $b^{-1}u$.

Let us put

$$C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}\begin{matrix} s_1 \\ s_2 \end{matrix}, \qquad f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}\begin{matrix} s_1 \\ s_2 \end{matrix}. \tag{13}$$

Without loss of generality we can assume that

$$\det C_2 b_2 \neq 0 \text{ on } \Gamma. \tag{14}$$

Let us consider

$$A = C_1(1 - b_2(C_2 b_2)^{-1} C_2) b_1. \tag{15}$$

We say that (1),(9) is a normal type problem if

$$\det A \neq 0 \ \text{ on } \Gamma. \tag{16}$$

Theorem 2. *Suppose that the conditions* (8), (10) *for the countour* Γ *and condition* (11) *for* C, f *are fullfilled. Then the problem* (1), (8) *is fredholmian in* $C^{1,\mu}(\overline{D})$ *if and only if the normality condition* (16) *is satisfied, and its index is*

$$æ = -\frac{1}{\pi} \arg\det A\Big|_{\Gamma} + s_1. \tag{17}$$

3 The case of the basic domain

Let the hyperbolic system (7) be such that the nilpotent part J_{20} of the matrix J_2 is equal to 0 and the diagonal matrix $\nu = (\nu_i \delta_{ij})$ is composed from two real numbers. Suppose that the boundary ∂D of the finite domain $D \subseteq \mathbb{C}$ consists of two noncharacteristic smooth curves Γ_1 and Γ_2 that connect two corner points τ_1 and τ_2. We consider the following boundary value problem

$$C_j u = f_j \text{ on } \Gamma_j, \quad j = 1, 2, \tag{18}$$

for the system (1), where C_j is a $(s_1 + s_2) \times l$ matrix and f_j is a $(s_1 + s_2)$ vector.

Let us introduce the weighted Holder space $C^{\mu}_{\lambda}(\overline{D}) = C^{\mu}_{\lambda}(\overline{D}; \tau_1, \tau_2)$, $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$, of all functions $\varphi(z)$ such that

$$\varphi(z) = |z - \tau_1|^{\lambda_1 - \mu} |z - \tau_2|^{\lambda_2 - \mu} \varphi_*(z),$$

where $\varphi_*(z) \in C^\mu(\overline{D})$ and $\varphi_*(\tau_1) = \varphi_*(\tau_2) = 0$. The classes $C^{1,\mu}_\lambda$ of differentiable functions are introduced by induction under the condition

$$\varphi \in C^\mu_\lambda, \quad \partial\varphi/\partial x_i \in C^\mu_{\lambda-1}.$$

We consider the problem (1),(18) in the class $C^{1,\mu}_\lambda(D)$ of solutions (1) such that the functions Φ and Ψ belong to this class in the representation (4).

Let $\gamma_j(t) \in C^{1,\mu}[0,1]$ be the smooth parametrization $[0,1] \to \Gamma_j$, $j = 1,2$ and the complex numbers $q_{2i-1} = \gamma'_i(0)$, $q_{2i} = -\gamma'_i(1)$ are the tangent vectors at the points τ_1, τ_2. By θ_j denote the angle of the sector corresponding to the corner τ_j. Evidently, $\theta_j = \arg q_k - \arg q_r$, $0 < \arg q < 2\pi$, $0 < \theta_j < 2\pi$, $P_j = \overline{k,r}$, $j = 1,2$, (more presicely, $P_1 = \overline{1,3}$, $P_2 = \overline{4,2}$), where the rotation from vector q_k to q_r about τ_j within domain is clock-wise.

Let us put the functions of matrices

$$m_j(\zeta) = (\mathrm{Re}\ q_k + (\mathrm{Im}\ q_k)J_1)^\zeta (\mathrm{Re}\ q_r + (\mathrm{Im}\ q_r)J_1)^{-\zeta}, \ \overline{k,r} = P_j,$$

and let be

$$\begin{aligned} x_1(\zeta) &= \left(e^{2\pi i\zeta} - 1\right)^{-1} \left(A_1(\tau_1)w_1(\zeta) + \overline{A_1(\tau_1)}v_1(\zeta)w_1(\zeta)\right), \\ x_2(\zeta) &= \left(e^{2\pi i\zeta} - 1\right)^{-1} \left(\overline{A_2(\tau_2)}v_2(\zeta)w_2(\zeta) + A_2(\tau_2)w_2(\zeta)\right), \end{aligned} \tag{19}$$

where

$$v_j = \begin{pmatrix} 0 & m_j(\zeta) \\ \overline{m_j^{-1}(\overline{\zeta})} & 0 \end{pmatrix}, \quad w_j(\zeta) = e^{2\pi i\zeta}\overline{v_j(\overline{\zeta})} - 1, \ j = 1,2,$$

$$A_j = c_{j,1}(1 - b_2(c_{j,2}b_2)^{-1}c_{j,2})b_1, \ j = 1,2.$$

Theorem 3. *Suppose that the conditiones* (8), (10) *for the curves* Γ_1 *and* Γ_2 *including the corner* τ_1, τ_2 *are fullfilled. Let* C, f *belong to* C^μ_λ *and the normality condition*

$$\det A_j(t) \neq 0, \quad t \in \Gamma_j, \ j = 1,2 \tag{20}$$

be satisfied.

Then the problem (1), (18) *is Fredholm in* $C^{1,\mu}_\lambda(\overline{D})$ *if and only if*

$$\det x_k(\zeta) \neq 0, \quad \mathrm{Re}\zeta = \lambda_k, \ k = 1,2, \tag{21}$$

and its index is

$$æ = -\frac{1}{\pi} \arg\det(A_1(t)A_2^{-1}(t))\Big|_0^1 - \frac{1}{2\pi}\sum_{k=1,2} \arg\det x_k(\zeta)\Big|_{\zeta=\lambda_k - i\infty}^{\lambda_k + i\infty} - s_1. \tag{22}$$

4 Some generalazations

We now consider the problem

$$C_j u = f_j \text{ on } \Gamma_j, \ j = 1,2, \tag{23}$$

in finite domains D, whose boundary ∂D consists of two curves Γ_1 and Γ_2 with the corners τ_1 and τ_2. We assume that the matrix $C_1(C_2)$ of order $(s_1 + s_2) \times l\ (s_1 \times l)$, and the vector $f_1(f_2)$

of order s_1+s_2 (s_1) are prescribed on $\Gamma_1(\Gamma_2)$ and f_j, C_j are belonged $H^k_{\mu,\lambda}$, $k=\overline{s}_2$, $j=1,2$. Here the curves Γ_j satisfy conditions (9), (10).

Theorem 4. *The assertion of theorem* 3 *remains in force also for the problem* (1), (23), *provided only that* A_j *mean matrices* $A_1 = C_{1,1}(1-b_2(C_{1,2}b_2)^{-1}C_{1,2})b_1$, $A_2 = C_2 b_1$, *and the last term* $-s_1$ *in the formula* (22) *must be replaced by the* s_1.

We also studied the questions of asymptotics of the solutions near the corner points and the smoothness of the solutions up to the boundary. We generalized this approach for the systems of higher order and for a class of the admissible finite domain with piecewise smooth boundary. If the order of C_1 in the last problem is not equal to s_1+s_2 then we investigated this problem only for the case $k_2=1$.

Our study is carried out in the framework of the function-theoretic method [5]. The scheme of this method is as follows. First of all we express a general regular solution in terms of regular solutions Φ and Ψ and use an anologue of a theorem of Vekua on integral representations of Φ and some notions about Ψ which arises from (7). By substituting that into the corresponding boundary conditions we reduce the problem to system of integral equations on the boundary of the domain. Another approuch see in [6].

Bibliography

1. J. Hadamard. Proprietes d'une equation line'aire aux derivees partielles du quatriemes ordre, Tohoky Math. Journ. – 1933.– v. 37. – P. 133 – 150.

2. L. Hormander. Linear partial differenrial operators. N.-Y.,1963.

3. N.A. Zhura. General boundary value problems for Douglis -Nirenberg elliptic systems in the domains with piese- wise smoothh boundary, Dokl. Akad. Nauk Russian 330 (1993), 151-154 (in Russian).

4. A.P. Soldatov. A function theory method in boundary value probles 1n the plane. I: The smooth case. Izv. Akad. Nauk SSSR Ser. Mat. 55 (1991), 1070-1100;(in Russian).

5. A.P. Soldatov. Singular integral operators and boundary value problems of functions theory. Vysh.Shkola, Moscow, 1991.

6. G.I. Eskin. Boundary value problems for equations with constant coefficients on the plane, Matem.sbornik Russian 59 (1962), 67-124 (in Russian).

СИСТЕМЫ УРАВНЕНИЙ ПЕРВОГО ПОРЯДКА ГЛАВНОГО ТИПА НА ПЛОСКОСТИ

Н.А. Жура

Физический институт имени П.Н.Лебедева РАН,

Ленинский проспект, 53, ФИАН, Москва, 119991, Россия, e-mail: nikzhura@gmail.com

Аннотация. В работе изучаются краевые задачи для систем уравнений первого порядка главного типа с постоянными коэффициентами на плоскости. При этом рассматриваются как полубесконечные области с нехарактеристической границей, так и конечные области типа луночки. Дано представление решений этих систем через решения более простых, так называемых, канонических систем первого порядка эллиптического и гиперболического типов. Получены также формулы для индекса соответствующих задач в весовых классах Гёльдера.

Ключевые слова: уравнения главного типа, нехарактеристическая граница, канонические системы первого порядка.

СВЕДЕНИЯ ОБ АВТОРАХ

А.Г. Александров	– доктор физико-математических наук, ведущий научный сотрудник Института проблем управления РАН (г. Москва, Россия)
Ксяо Линг Ванг	– профессор Шаосинского колледжа искусства и науки (г. Шаосин, Китай)
Янг Пинг Ванг	– профессор Шаосинского колледжа искусства и науки (г. Шаосин, Китай)
В.И. Власов	- доктор физико-математических наук, профессор, заведующий сектором аналитико-численных методов математической физики Вычислительного центра им. А. А. Дородницына РАН (г. Москва, Россия)
В.А. Грузман	– кандидат технических наук, заведующий лабораторией гипертекстовых систем представления знаний Института проблем управления РАН (г. Москва, Россия)
Л.А. Гутник	– кандидат физико-математических наук, доцент Московского государственного института электроники и математики (г. Москва, Россия)
Н.А. Жура	– кандидат физико-математических наук, старший научный сотрудник Физического института им. П.Н.Лебедева РАН (г. Москва, Россия)
А.А. Кастро мл.	– профессор Федерального университета штата Баия (г. Баия, Бразилия)
Жан Фенг Ксю	– профессор Вууй университета (г. Гуандонг, Китай)
Фенг Лю	– профессор Китайского нефтяного университета (г. Шаньдун, Китай)
Тимур Садыков	– кандидат физико-математических наук, доцент факультета математики и информатики Сибирского федерального университета (г. Красноярск, Россия)
Джиро Секигучи	– профессор Токийского университета сельского хозяйства и технологии (г. Токио, Япония)
С.Л. Скороходов	– кандидат физико-математических наук, старший научный сотрудник Вычислительного центра им. А. А. Дородницына РАН (г. Москва, Россия)
А.П. Солдатов	– доктор физико-математических наук, профессор, заведующий кафедрой математического анализа Белгородского государственного университета (г. Белгород, Россия)
Пей Чу Ху	– профессор Шаньдунского университета (г. Шаньдун, Китай)
Линг Ди Хуанг	– профессор Шаосинского колледжа искусства и науки (г. Шаосин, Китай)
Чунг Чун Янг	– профессор Гонконгского университета науки и технологии (р-н Коулун, Гонконг)

ИНФОРМАЦИЯ ДЛЯ АВТОРОВ

Принимаются рукописи статей, написанные на русском (или на английском) языке, по различным разделам математики и физики. Содержание статей может содержать как результаты оригинальных исследований автора (ов), так и представлять собой обзор по выбранной автором (ами) теме.

Статья должна быть написана с достаточной степенью подробности с таким расчётом, чтобы быть понятной не только узким специалистам по выбранному автором (ами) направлению исследований, но более широкому кругу, соответственно, математиков и физиков. Ни в коем случае, рукопись не должна представлять собой краткий отчёт о проведенных исследованиях, написанный в виде краткого сообщения, не содержащий описания постановки задачи (соответственно, условий проведения эксперимента). В связи с этим, рукопись должна быть структурирована – разделена на разделы, представляющие отдельные смысловые единицы текста. В любом случае, рукопись должна содержать введение и заключение.

Во введении должна быть кратко описана проблема, которой посвящена рукопись, определено место этой проблемы в общем объёме физико-математического знания, должна быть дана краткая история вопроса и описан полученный автором (ами) результат. В заключении должна быть дана краткая характеристика полученного результата и указано его значение в дальнейшем развитии темы работы.

Те же самые требования к введению и заключению предъявляются для обзорной статьи, с той лишь разницей, что её содержание должно быть посвящено описанию всей совокупности результатов, отражающих состояние выбранной автором области исследований, и сам текст должен быть написан с большей степенью подробности.

Принимаются также для публикации статьи, носящие методический характер. Но в этом случае решение о возможности публикации такой рукописи принимается отдельным решением редколлегии журнала.

Рукопись должна быть оформлена в соответствии с традициями написания, соответственно, математических и физических текстов. В частности, в математических текстах должны быть чётко выделены такие структурные единицы, как формулировки определений, теорем и лемм, следствий и замечаний, отмечены начала и окончания доказательств.

Полный объём рукописи, которая представляет собой оригинальное исследование, не должен превышать 20 страниц формата А4. Она должна быть написана шрифтом 14pt через два интервала. Объём обзорной статьи необходимо заранее оговорить с редколлегией журнала.

Рукопись должна состоять из следующих частей:

1) основной содержательной части, представляемой на русском или английском языках. Она должна начинаться указанием номера УДК того научного направления, которому посвящена статья. Затем следует название статьи. Оно должно состоять не более чем из 20 слов. Далее приводится список авторов статьи, затем следует полностью основная часть рукописи;

2) аннотации на русском языке. Её объём не должен превышать 10-12 строк, написанных шрифтом 12pt;

3) списка ключевых слов (не более 10-12).

4) перевода заглавия, аннотации и ключевых слов на английский язык;

5) списка литературных источников, на которые имеются ссылки в тексте рукописи;

6) данных об авторах статьи с указанием места работы, точного почтового адреса и занимаемой должности. Должны быть указаны адреса электронной почты. Эти данные необходимо представить также на английском языке. Кроме того, должна быть дана латинская транскрипция фамилий авторов. Соответственно, для статей на английском языке, должна быть дана транскрипция фамилий авторов кириллицей;

7) списка подписей к рисункам, если они имеются в рукописи;

8) укороченного заголовка статьи, состоящего не более, чем из трёх слов, который печатается в колонтитулах журнала.

В редакцию присылается электронный вариант рукописи. Он должен быть подготовлен в редакторе LaTeX (LaTeX2e, AMSLaTeX). При этом нужно также прислать файл с pdf-копией рукописи для того, чтобы редакция имела возможность сравнения с авторским оригиналом при редактировании.

Если в рукописи имеются рисунки, то они должны быть подготовлены в формате "eps" и соответствующие им файлы необходимо пронумеровать в соответствии со списком подписей к рисункам (см. п.7).

Особые требования к электронному набору в редакторе LaTeX следующие:

1) нельзя использовать вводимые авторами новые нестандартные команды;

2) выключные формулы должны быть пронумерованы в порядке их появления в рукописи в том случае, если на них есть ссылки в тексте. При использовании режима equation для набора выключных формул обязательно употребление для их нумерации цифровых меток, соответствующих номеру формулы. Допускается применение для нумерации формул цифр, снабжённых штрихами. Однако, этим нужно пользоваться только в случае крайней необходимости с целью более точной передачи смысла текста. В случае, если в статье имеются части в виде приложений, нумерация содержащихся в них выключных формул может быть не зависимой от нумерации основного текста. При этом в приложениях рекомендуется употребление двойной нумерации, в которой первый символ может быть прописной буквой или номером приложения;

3) то же самое касается литературных источников, на которые имеются ссылки в тексте рукописи. Их нужно отмечать цифрами в порядке появления в тексте, и ни в коем случае не использовать метки другого типа.